PAPER

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Initial states of qubit–environment models leading to conserved quantities

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Abstract

It is possible to prepare a composite qubit–environment system so that its time evolution will guarantee the conservation of a preselected qubit's observable. In general, this observable is not associated with a symmetry. The latter may not even be present in the subsystem. The initial states which lead to such a quantity conserved dynamics form a subspace of the qubit–environment space of states. General construction of this subspace is presented and illustrated by two examples. The first one is the exactly solvable Jaynes–Cummings model and the second is the multi-photon Rabi model.

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1. Introduction

An interaction between two quantum systems almost always modifies a set of quantum numbers suitable for the description of non-interacting components. Generically, only in the absence of interactions does the set of good quantum numbers of a composite system consist of good quantum numbers of its subsystems. Let us consider energy as an example. It is well known that (Hamiltonian-type) interactions cause an energy flow from one system to the other and/or vice-versa. The total system remains conservative but the interacting parts become *open*, which changes their character qualitatively [1].

One can name at least one case, commonly referred to as *dephasing* [2], where the energy exchange between two systems is absent during the entire evolution regardless of the initial conditions. In open quantum system theory, where it is often assumed that the second subsystem is much larger than the first one, this phenomenon is also known as pure decoherence [3]. It induces the loss of quantum coherence (and, as a result, the emergence of classical behaviour [4]) without affecting the system's energy.

For dephasing to occur a rather specific type of the interaction is required, yet nothing which might contradict our intuition is behind this phenomenon. The energy lossless evolution is due to the existence of a conservation law: the total Hamiltonian commutes with the interaction operator. It is also the case if, instead of the energy, one considers a quantum

number related to a local observable (i.e. acting non-trivially only on a subsystem). In general, one cannot expect such a number to be conserved unless the corresponding observable happens to be a symmetry in that subsystem.

One can ask: is it possible to design such a time evolution of a composite quantum system that makes a local quantum number a good quantum number regardless of the existence of the related symmetry in the subsystem? This is very intriguing as we are really asking whether or not the conservation of a quantum number in an open system is possible, not in spite of the destructive influence of its environment but rather because of it. At face value, this may sound paradoxical (to say the least), yet similar behaviour has been shown to take place in e.g. quantum Markovian systems [5].

Such a behaviour might serve as a model of a different type of computation—dissipative quantum computation, in which it is the environment which does all the work. Instead of isolating a system (and then applying a tractable unitary dynamics on it), as in the standard paradigm of quantum computation [6], we engineer its coupling with the environment to perform the desired tasks [7, 8] (as in thermodynamics [9]). In a scenario like this, the interaction is described by a Markovian master equation [10]. The computation task is encoded in Lindblad operators, which model the environment, whereas the unique outcome (the result of the computation) is written into the steady state reached rapidly by the open system.

This work is devoted to showing that the destructive nature of the environment can also be explored to induce conservation of information encoded in open quantum systems. We only consider two-level systems; the general case remains open. There are many potential applications of this mechanism. Take for example quantum information science.

Quantum devices of the future, such as e.g. quantum computers [11], will most likely be composed of components which consist of a large amount of qubits (quantum memory [12], quantum register [13], etc). The challenge is not only to stabilize such devices (shield against decoherence) but also to program them. For instance, for various reasons (e.g. to establish a reference point or to cache the results) it may be desirable to freeze the spin of a single qubit (or a cluster of them) in a given (preselected) direction during computational cycles. In principle, this could be achieved by means of quantum control, usually via suitable drivings. Instead of (either open loop or feedback) quantum control, which continuously affects the system being analysed, we propose a method in which it is sufficient to measure the system only once at beginning of the evolution (initial preparation). No dynamical control is needed to maintain the desired dynamics. Such an alternative approach may be more beneficial as the goal can be achieved without introducing 'numerical' errors resulting from the measurement. Of course, to fully explore this idea in real environments much work, both experimental and theoretical, needs to be done. We hope that this paper will serve as a simple theoretical starting point.

The layout of this paper is as follows. We begin by showing how a proper choice of the initial state of a composite qubit–boson system can assure no energy exchange between the two subsystems. In other words, we identify dephasing-like behaviour in a non-dephasing model. Next, in section 3, we generalize this idea further, beyond the energy, by showing how to prepare a qubit–environment system so that the information encoded in a preselected qubit observable cannot be erased by its environment. Section 4, followed by the conclusions (section 5), serves as our second example.

2. An example: conservation of energy

Let us start with the Jaynes–Cummings (JC) model [14],

$$H = \omega\sigma_z + \nu a^\dagger a + (g^*\sigma_+ \otimes a + g\sigma_- \otimes a^\dagger). \tag{1}$$

a and $a^\dagger$ are the creation and annihilation operators of the bosonic field (environment) i.e., $[a, a^\dagger] = \mathbb{I}_B$, whereas σ_z denotes one of the Pauli spin matrices with eigenstates $|\pm\rangle$. $\sigma_\pm$ are ladder operators, $\sigma_\pm|\mp\rangle = |\pm\rangle$ and $\sigma_\pm|\pm\rangle = 0$. ν and ω describe energies of the qubit and the field, respectively. The coupling constant g reflects the strength of the interaction between the systems.

Although the model (1) essentially originates from quantum optics [14], it has been studied in a wide range of other branches of physics (see e.g.[15]) for almost half a century within a broad variety of contexts [16]. Here, it serves as a good starting point for our considerations.

In this example, we are interested in finding a qubit–boson initial state, ϱ, such that the qubit energy, $E_Q = \omega \mathrm{Tr}(\sigma_z \otimes \mathbb{I}_B \varrho(t)) \sim \langle \sigma_z \rangle$, is conserved. In other words, we wish to have $E_Q = \omega \mathrm{Tr}(\sigma_z \rho(t)) = \mathrm{cst}$. where $\rho(t) = \mathrm{Tr}_B(\mathrm{e}^{-\mathrm{i}Ht} \varrho \mathrm{e}^{\mathrm{i}Ht})$ stands for the reduced qubit dynamics [10].

It is well known that for the JC model the total number of excitations, $N = a^\dagger a + \sigma_z$, is conserved. This fact ultimately leads to the exact diagonalization of the JC Hamiltonian (the exact dynamics of the JC model is known). At this point, we want to emphasize that none of the above features is essential in the following analysis, yet involving N one simplifies calculations.

We start by splitting the Hamiltonian (1) into two commuting parts, $H = H_0 + V$, where $H_0 = \nu N$ and

$$V = \delta \sigma_z + (g^* \sigma_+ \otimes a + g \sigma_- \otimes a^\dagger). \tag{2}$$

$\delta \equiv \omega - \nu$ is the detuning frequency. As a result, the time evolution of the system can be factorized such that

$$U_t = \exp(-\mathrm{i}H_0 t) \exp(-\mathrm{i}Vt) \equiv U_t^0 V_t. \tag{3}$$

Note that $\rho(t)$ mimics a pure dephasing evolution if

$$\rho(t) = \alpha |+\rangle\langle +| + c(t)^* |-\rangle\langle +| + c(t) |+\rangle\langle -| + (1-\alpha)|-\rangle\langle -|, \tag{4}$$

where α is a real constant and $c(t)$ denotes a function of time such that $|c(t)| \leqslant 1$. So, the questions are: which initial states of the composite system do guarantee the reduced dynamics (4) and are they separable or entangled?

In order to give an answer to these questions, we offer an explicit construction of such states. At face value, subsequent steps may seem bizarre and to some extent artificial. Notwithstanding this, they can be performed, as is discussed below, for a broad class of qubit–boson models. As a first step, we define $\varrho = |\Psi\rangle\langle\Psi|$ where

$$|\Psi\rangle = C_\psi (|+\rangle \otimes |\psi\rangle + |-\rangle \otimes X|\psi\rangle), \tag{5}$$

and $|\psi\rangle$ is an arbitrary state of the bosonic field. C_ψ is a normalization constant chosen so that $\langle\Psi|\Psi\rangle = 1$. For the sake of simplicity, we absorb it into $|\psi\rangle$ so that $\|\psi\| = |C_\psi|$.

A possibility of constructing dephasing states relies on the ability to find a linear operator X satisfying

$$g^* X a X + 2\delta X - g a^\dagger = 0, \tag{6}$$

or $X(\delta + g^* a X) = -\delta X + g a^\dagger$. As it will be seen, this choice of X allows for a specific time evolution of the above state, namely

$$|\Psi\rangle \to |\Psi_t\rangle = U_t^0 (|+\rangle \otimes |\psi_t\rangle + |-\rangle \otimes X|\psi_t\rangle). \tag{7}$$

According to equation (7), the dynamics of the entire system consists of two parts. The one governed by $U_t^0 = \exp(-\mathrm{i}H_0 t)$ is the free evolution. The other one, encoded in $|\psi_t\rangle$, is the

remaining part resulting from the interaction. It needs to be determined along with X. As a result of the evolution (7), we obtain the following reduced dynamics

$$\alpha(t) = \langle\psi_t|\psi_t\rangle \quad \text{and} \quad c(t) = \mathrm{e}^{-\mathrm{i}2\nu t}\langle\psi_t|\mathrm{e}^{\mathrm{i}\nu a^\dagger a t} X \mathrm{e}^{-\mathrm{i}\nu a^\dagger a t}|\psi_t\rangle, \tag{8}$$

which resembles the dephasing dynamics (4) if the vector $|\psi_t\rangle$ itself evolves unitarily. For this to hold true there must exist a Kamiltonian K such that $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}Kt}|\psi\rangle$. To see if this is indeed the case here let us assume that $K = \delta + g^* aX$. Then it follows from (6) and (2) that

$$\begin{aligned} V|\Psi\rangle &= |+\rangle \otimes K|\psi\rangle + |-\rangle \otimes (-\delta X + g a^\dagger)|\psi\rangle \\ &= |+\rangle \otimes K|\psi\rangle + |-\rangle \otimes XK|\psi\rangle, \end{aligned} \tag{9}$$

which holds for every $|\psi\rangle$. Thus, by replacing $|\psi\rangle$ with $K|\psi\rangle$ one finds

$$V^2|\Psi\rangle = |+\rangle \otimes K^2|\psi\rangle + |-\rangle \otimes XK^2|\psi\rangle. \tag{10}$$

Repeating this step n times we have

$$V^n|\Psi\rangle = |+\rangle \otimes K^n|\psi\rangle + |-\rangle \otimes XK^n|\psi\rangle \tag{11}$$

and finally

$$V_t|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!} V^n|\Psi\rangle = |+\rangle \otimes \mathrm{e}^{-\mathrm{i}Kt}|\psi\rangle + |-\rangle \otimes X\mathrm{e}^{-\mathrm{i}Kt}|\psi\rangle. \tag{12}$$

It remains 'only' to identify X and to show that K is Hermitian. There is no general method allowing us to find solutions of the quadratic equations (6), yet judging from its structure it seems reasonable to try $X = \sum_n \xi_n |n+1\rangle\langle n|$. One can infer that this is indeed a good guess as long as

$$\xi_n = \frac{-\delta + \sqrt{\delta^2 + |g|^2(n+1)}}{g^*\sqrt{n+1}}, \quad n \geqslant 0. \tag{13}$$

The solution we have just found generalizes the Susskind–Glogower operator [17] in the sense that $X \to (aa^\dagger)^{-1/2}a^\dagger$ when $\delta \to 0$. As a result, $K = \sqrt{\delta^2 + |g|^2}(a^\dagger a + \mathbb{I}_B)$ and is clearly Hermitian. It follows from equation (8) that $\alpha = \langle\psi|\psi\rangle$ and in addition

$$c(t) = \mathrm{e}^{-\mathrm{i}\nu t} \sum_{n=0}^{\infty} \xi_n \mathrm{e}^{\mathrm{i}\Omega_n t}\langle\psi|n+1\rangle\langle n|\psi\rangle, \tag{14}$$

where $\Omega_n = \sqrt{\delta^2 + |g|^2(n+2)} - \sqrt{\delta^2 + |g|^2(n+1)}$.

Note that we still have the freedom to choose the vector $|\psi\rangle$. In particular, putting $|\psi\rangle \sim |m\rangle$, where $|m\rangle$ is a state with a defined number of bosons, we not only have $\dot{c}(t) = 0$ (meaning $\rho(t)$ is a steady state) but also $c(t) = 0$ (i.e. the state exhibits its classical nature).

In general, states such as the one from equation (5) are entangled. Hence, there is no one-to-one correspondence between qubit–boson density operators $|\Psi\rangle\langle\Psi|$ and reduced density matrices $\mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$ of the qubit [18]. If, however, $|\psi\rangle$ is an eigenvector of X, that is $X|\psi\rangle = \lambda|\psi\rangle$ for $\lambda \in \mathbb{C}$, then $|\Psi\rangle = (|+\rangle + \lambda|-\rangle) \otimes |\psi\rangle$ is separable.

3. Generalization

So far, we have shown how to determine the initial preparations of a (specific) composite qubit–boson system which guarantees no energy flow between the qubit and its environment (boson). In that case, the reduced qubit dynamics mimics a pure dephasing evolution. We have also investigated conditions under which from this broad class of states one can choose separable ones.

Now, we are interested in finding an answer to a more general question: how does one prepare an initial state of a composite system (and determine its separability) which results in no 'information flow' between its subsystems? We assume that the information is encoded in a qubit observable Λ. As we will see, such states have very much in common with dephasing states and for that reason we will keep this terminology.

In section 2, we have conducted our analysis by making use of a very specific exactly solvable model. Currently, we will show that neither solvability nor the form of the interaction in this model is a *sine qua non* condition for building 'dephasing states' for general qubit–environment models.

Let Λ be a 2×2 Hermitian matrix—a given qubit observable. Our objective is to determine $\rho \equiv \rho(0)$ such that $\langle \Lambda(t)\rangle \equiv \mathrm{Tr}(\Lambda\rho(t))$ remains constant during the evolution. As before, by $\rho(t)$ we denote the qubit reduced dynamics, $\rho(t) = \mathrm{Tr}_E[\mathrm{e}^{-\mathrm{i}\mathbf{H}t}|\Psi\rangle\langle\Psi|\mathrm{e}^{\mathrm{i}\mathbf{H}t}]$, where $|\Psi\rangle$ is the initial qubit–environment state and

$$\mathbf{H} = H_Q \otimes \mathbb{I}_E + \mathbb{I}_Q \otimes H_E + \mathbf{H}_{\mathrm{int}}. \tag{15}$$

with all the symbols having their usual meaning, stands for the total Hamiltonian.

We begin with a very simple observation that in each moment of time and for every complex 2×2 matrix Λ the partial trace, $\mathrm{Tr}_E(\cdot)$, satisfies

$$\mathrm{Tr}[\Lambda \mathrm{Tr}_E(\varrho(t))] = \mathrm{Tr}[(\Lambda \otimes \mathbb{I}_E)|\Psi(t)\rangle\langle\Psi(t)|]. \tag{16}$$

By virtue of this relation, we have

$$\langle \Lambda(t)\rangle = \mathrm{Tr}[(\Lambda_d \otimes \mathbb{I}_E)|\Omega(t)\rangle\langle\Omega(t)|], \tag{17}$$

where $|\Omega(t)\rangle = \mathrm{e}^{-\mathrm{i}\mathbf{K}t}|\Omega\rangle$ with $|\Omega\rangle = U \otimes \mathbb{I}_E|\Psi\rangle$ and

$$\mathbf{K} = (U \otimes \mathbb{I}_E)\mathbf{H}(U \otimes \mathbb{I}_E)^\dagger. \tag{18}$$

U denotes the unitary matrix such that $U^\dagger \Lambda U \equiv \Lambda_d = \mathrm{diag}(\lambda_+, \lambda_-)$.

The Kamiltonian $\mathbf{K}$ can always be written as

$$\mathbf{K} = |+\rangle\langle+| \otimes H_+ + |-\rangle\langle-| \otimes H_- + |+\rangle\langle-| \otimes V + |-\rangle\langle+| \otimes V^\dagger. \tag{19}$$

An explicit form of $H_\pm$ and V can easily be recovered when the Hamiltonians H_Q, H_E, and $\mathbf{H}_{\mathrm{int}}$ are provided. *A priori*, we neither impose any physical restriction of their specification nor assume the existence of symmetries in the total system.

The composite system is assumed to be in the state $\varrho = |\Psi\rangle\langle\Psi|$ initially, where

$$|\Psi\rangle = C_\psi(|\lambda_+\rangle \otimes |\psi\rangle + |\lambda_-\rangle \otimes X|\psi\rangle), \tag{20}$$

with $|\lambda_\pm\rangle = U^\dagger|\pm\rangle$. $|\psi\rangle$ is a freely chosen state of the environment. As before, one can redefine the state $|\psi\rangle$ so that $|\psi\rangle \to C_\psi|\psi\rangle$. We show that $|\Omega\rangle$ undergoes the following evolution

$$|\Omega\rangle \to \mathrm{e}^{-\mathrm{i}\mathbf{K}t}|\Omega\rangle = |+\rangle \otimes |\psi_t\rangle + |-\rangle \otimes X|\psi_t\rangle, \tag{21}$$

where $|\psi_t\rangle = \mathrm{e}^{-\mathrm{i}K_+t}|\psi\rangle$ for some operator K_+, provided that X solves the (operator) Riccati equation,

$$XVX + XH_+ - H_-X - V^\dagger = 0. \tag{22}$$

K_+ and thus $|\psi_t\rangle$ are yet to be determined. Riccati equations of the type (22) have not only been studied in mathematical contexts [19] but have also recently been applied to the investigation of open quantum systems [20]. As one may anticipate, equation (22) reduces to the condition (6) when $\mathbf{H}$ is given by (1).

It immediately follows from (22) that $X(H_+ + VX) = H_-X + V^\dagger$ which leads to

$$\mathbf{K}|\Omega\rangle = |+\rangle \otimes K_+|\psi\rangle + |-\rangle \otimes XK_+|\psi\rangle \quad \text{where} \quad K_+ = H_+ + VX. \tag{23}$$

As easily as before, one can justify the general formula,

$$\mathbf{K}^n|\Omega\rangle = |+\rangle \otimes K_+^n|\psi\rangle + |-\rangle \otimes XK_+^n|\psi\rangle \quad (n \geqslant 0). \tag{24}$$

As a result, the evolution generated by $\mathbf{K}$ reads

$$|\Omega(t)\rangle = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!}\mathbf{K}^n|\Psi\rangle = |+\rangle \otimes \mathrm{e}^{-\mathrm{i}K_+t}|\psi\rangle + |-\rangle \otimes X\mathrm{e}^{-\mathrm{i}K_+t}|\psi\rangle. \tag{25}$$

Having (25) in place, we can begin to investigate conditions upon which the expectation value (17) does not depend on time. Except for the fact that α is time-dependent, $\mathrm{Tr}_\mathrm{E}(|\Omega(t)\rangle\langle\Omega(t)|)$ has a structure which is similar to the dephasing matrix (4). To be more specific,

$$\alpha(t) = \langle\psi|\mathrm{e}^{\mathrm{i}K_+^\dagger t}\mathrm{e}^{-\mathrm{i}K_+t}|\psi\rangle \quad \text{and} \quad c(t) = \langle\psi|\mathrm{e}^{\mathrm{i}K_+^\dagger t}X\mathrm{e}^{-\mathrm{i}K_+t}|\psi\rangle. \tag{26}$$

These formulas generalize those given in equation (8) as one would expect.

Similar dynamics can also be designed by starting from any state orthogonal to (20). Indeed, such orthogonal states are found to be of the form

$$|\Phi\rangle = C_\phi(|\lambda_-\rangle \otimes |\phi\rangle - |\lambda_+\rangle \otimes X^\dagger|\phi\rangle), \tag{27}$$

which can be verified by noticing that $\langle\Psi|\Phi\rangle = 0$ for every $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}_E$. In this case, the qubit's reduced dynamics takes the form

$$\alpha(t) = 1 - \langle\phi_t|\phi_t\rangle, \quad c(t) = -\langle\phi_t|X^\dagger|\phi_t\rangle, \quad |\phi_t\rangle = \mathrm{e}^{-\mathrm{i}K_-t}|\phi\rangle. \tag{28}$$

where $K_- = H_- - V^\dagger X^\dagger$. For instance, in the JC model discussed in the previous section we have $K_- = \sqrt{\delta^2 + |g|^2 a^\dagger a}$.

Henceforward, we assume that the composite system is prepared initially in one of the states (4). Then it follows from equation (17) that

$$\langle\Lambda(t)\rangle = \alpha(t)\lambda_+ + (1 - \alpha(t))\lambda_-. \tag{29}$$

Clearly, if $\alpha(t)$ is a constant so is $\langle\Lambda(t)\rangle$. Thus the question is: upon which conditions can we have $\alpha(t) = \mathrm{cst.}$? Obviously, it depends on the particular properties of K_+. So, what are they? First of all, if VX is Hermitian, as in our opening example, so is K_+ and hence the evolution $|\psi\rangle \to |\psi_t\rangle$ is unitary. Thus $\alpha(t) = \alpha(0)$ and yet $c(t) \neq c(0)$ which results in not trivial dephasing dynamics regardless of the initial vector $|\psi\rangle$.

The really interesting question here is: what about all the cases when K_+ is not Hermitian? Are those even possible? It is hard to address these questions since we have no knowledge regarding X. Nevertheless, with a fair amount of intuition combined with justifiable assumptions concerning the operator K_+ we can overcome this problem.

First, we argue that $\mathrm{K}_\pm$ are pseudo-Hermitian i.e., there are some invertible η and ξ such that $\mathrm{K}_+^\dagger = \eta\mathrm{K}_+\eta^{-1}$ and $\mathrm{K}_-^\dagger = \xi\mathrm{K}_-\xi^{-1}$ [21]. Indeed, if X solves the Riccati equation (22) then (note $\mathbb{C}^2 \otimes \mathcal{H}_\mathrm{E} = \mathcal{H}_\mathrm{E} \oplus \mathcal{H}_\mathrm{E}$)

$$\begin{bmatrix} 1 & -\mathrm{X}^* \\ \mathrm{X} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{H}_+ & \mathrm{V} \\ \mathrm{V}^\dagger & \mathrm{H}_- \end{bmatrix} \begin{bmatrix} 1 & -\mathrm{X}^* \\ \mathrm{X} & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{K}_+ & 0 \\ 0 & \mathrm{K}_- \end{bmatrix}, \tag{30}$$

meaning, in particular, that $\sigma(\mathbf{K}) = \sigma(\mathrm{K}_+) \cup \sigma(\mathrm{K}_-)$. Moreover, if one defines $\mathbf{U}$ to be the unitary matrix from the polar decomposition of $\mathbf{S}$, $\mathbf{S} = \mathbf{U}\sqrt{\mathbf{S}^\dagger\mathbf{S}}$, then it follows from (30) that

$$\mathbf{U}^\dagger \begin{bmatrix} \mathrm{H}_+ & \mathrm{V} \\ \mathrm{V}^\dagger & \mathrm{H}_- \end{bmatrix} \mathbf{U} = (\mathbf{S}^\dagger\mathbf{S})^{1/2} \begin{bmatrix} \mathrm{K}_+ & 0 \\ 0 & \mathrm{K}_- \end{bmatrix} (\mathbf{S}^\dagger\mathbf{S})^{-1/2}. \tag{31}$$

Since a unitary transformation preserves Hermiticity, the latter equality proves the pseudo-Hermiticity conditions given above for $\eta = \mathbb{I}_\mathrm{E} + \mathrm{X}^\dagger\mathrm{X}$ and $\xi = \mathbb{I}_\mathrm{E} + \mathrm{X}\mathrm{X}^\dagger$, respectively. It should

be obvious that η and ξ induce positive definite inner products $\langle \eta \cdot, \cdot \rangle$ and $\langle \xi \cdot, \cdot \rangle$ (because of which they are called metric operators) with respect to which K_+ and K_- are Hermitian.

It seams reasonable to assume that the Kamiltonian K_+ has a complete set of eigenstates. For the sake of argument, let us also assume that its spectrum is discrete and not degenerated (i.e. $\mathrm{K}_+|\psi_n\rangle = E_n|\psi_n\rangle$). Then

$$\mathrm{K}_+ = \sum_n E_n|\psi_n\rangle\langle\psi_n| \quad \text{and} \quad \mathrm{K}_+^\dagger = \sum_n E_n^*|\psi_n\rangle\langle\psi_n|. \tag{32}$$

Now, the interesting part follows. According to (30), an eigenvalue E_n of K_+ is also an eigenvalue of $\mathbf{K}$. Thus $E_n^* = E_n$ proving that K_+ is in fact Hermitian.

If one cannot find a basis which consists with eigenstates of the Kamiltonian K_+ we still can design a dephasing dynamics as long as K_+ is diagonalizable. This condition is much weaker than Hermiticity and seems to reflect an absolute minimal physical requirement that one can impose on K_+ in this context [22].

As before, we only investigate the discrete and not the degenerated case. Saying that K_+ can be diagonalized means that there is a basis $\{|n\rangle\}$, linear invertible transformation S and complex numbers E_n (which we know are in fact real) such that

$$\mathrm{S}^{-1}\mathrm{K}_+\mathrm{S} = \sum_n E_n|n\rangle\langle n|. \tag{33}$$

Now, we can introduce two sets of vectors: $|\psi_n\rangle = \mathrm{S}|n\rangle$ and $|\phi_n\rangle = (\mathrm{S}^{-1})^\dagger|n\rangle$ which form a complete set of biorthonormal eigenvectors [23], that is

$$\mathrm{K}_+|\psi_n\rangle = E_n|\psi_n\rangle, \quad \mathrm{K}_+^\dagger|\phi_n\rangle = E_n|\phi_n\rangle, \quad \mathbb{I}_\mathrm{E} = \sum_n |\psi_n\rangle\langle\phi_n|, \quad \langle\psi_n|\phi_m\rangle = \delta_{nm}. \tag{34}$$

At this point, it should be stressed that $\langle\psi_n|\psi_m\rangle \neq \delta_{nm}$ and $\langle\phi_n|\phi_m\rangle \neq \delta_{nm}$ in general. In view of (34), we have

$$\mathrm{K}_+ = \sum_n E_n|\psi_n\rangle\langle\phi_n| \quad \text{and} \quad \mathrm{K}_+^\dagger = \sum_n E_n|\phi_n\rangle\langle\psi_n|. \tag{35}$$

Note that if the similarity operator S is unitary, so is the evolution $|\psi\rangle \to |\psi_t\rangle$ since the two bases $\{|\psi_n\rangle\}$,$\{|\psi_m\rangle\}$ are identical and thus K_+ is Hermitian. Henceforward, we assume this is not the case because we have already examined it. From both (26) and (34) we have

$$\alpha(t) = \sum_n |\langle\psi|\phi_n\rangle|^2\|\psi_n\|^2 + \sum_{n\neq m} \mathrm{e}^{\mathrm{i}(E_n-E_m)t}\langle\psi|\phi_n\rangle\langle\psi|\phi_m\rangle^*\langle\psi_n|\psi_m\rangle \tag{36}$$

and in addition

$$c(t) = \sum_n |\langle\psi|\phi_n\rangle|^2\|\mathrm{X}\psi_n\|^2 + \sum_{n\neq m} \mathrm{e}^{\mathrm{i}(E_n-E_m)t}\langle\psi|\phi_n\rangle\langle\psi|\phi_m\rangle^*\langle\psi_n|\mathrm{X}|\psi_m\rangle. \tag{37}$$

Both these expressions are time-dependent as they should be (in a Hilbert space one cannot have a linear and non-unitary evolution which preserves the norm). We can, however, make $\alpha(t)$ constant at least for certain initial states $|\psi\rangle$. In particular, if $|\psi\rangle = |\psi_n\rangle$ then both $\alpha(t)$ and $c(t)$ does not depend on time. The latter observation is in perfect agreement with the results regarding the stationary states reported in [24].

4. Second example

As a second example, we consider the k-photon Rabi model [25] for which $H_Q = \omega\sigma_z$ and $H_E = \nu a^\dagger a$, which is read exactly as in the JC model (1) but the interaction is given by

$$\mathbf{H}_\mathrm{int} = \sigma_x \otimes (g^* a^k + g(a^\dagger)^k). \tag{38}$$

This model not only generalizes the single mode case but also includes counter-rotating-wave terms [26], which make its analytical treatment much more complicated in comparison with (1). Despite recent progress in finding its analytic solution [27], the problem remains highly non-trivial.

Let us suppose that this time we want to find initial state(s) of the composite system such that the x-component of the qubit spin operator remains constant during the evolution, $S_x = \frac{1}{2}\mathrm{Tr}(\sigma_x \rho(t)) = \mathrm{cst}$.

First, one needs to determine U which transforms σ_x to its diagonal form. This is an easy task to do and the answer is $U = (\sigma_z + \sigma_x)/\sqrt{2}$. In this case, we have $U\sigma_x U^\dagger = \sigma_z$, that is $\lambda_\pm = \pm 1$. Next, we transform $\mathbf{H}$ into $\mathbf{K}$ according to (18) and then we recover $H_\pm, V$ from the decomposition (19). Straightforward calculation shows

$$H_\pm = \nu a^\dagger a \pm (g^* a^k + g(a^\dagger)^k), \quad V = \omega \mathbb{I}_E. \tag{39}$$

The corresponding Riccati equation (22) reads as follows

$$\omega X^2 + X H_+ - H_- X - \omega \mathbb{I}_E = 0. \tag{40}$$

We can solve this equation by introducing the generalized parity operator [28]:

$$X_k = \sum_{l=1}^{k} \sum_{n=0}^{\infty} (-1)^n |n, l\rangle\langle n, l|, \tag{41}$$

where $|n, l\rangle := |kn + l - 1\rangle$ and $l \leqslant k$. This operator is both Hermitian and unitary (in particular $X_k^2 = \mathbb{I}_E$). For $k = 1$ it simplifies to the well-known bosonic parity $P = \exp(\mathrm{i}\pi a^\dagger a)$. As can be verified, $X_k a^k X_k = -a^k$ and thus $X_k H_+ X_k = H_-$. Therefore, X_k indeed solves (40). This may come as a surprise since (41) does not depend on any of the parameters ν, ω, g.

In this model, the dephasing dynamics (26) and (28) are generated by $K_\pm = H_\pm \pm \omega X_k$, respectively. Introducing projections $P_\pm = \frac{1}{2}(\mathbb{I}_E \pm X_k)$ onto subspaces $\mathcal{H}_\pm$ consisting of states with defined parity (with respect to the generalized parity (41)) and taking into account both (5) and (27), we have

$$|\Psi_\epsilon\rangle = \tfrac{1}{2}(|+\rangle \otimes P_\epsilon|\psi\rangle + |-\rangle \otimes P_{-\epsilon}|\psi\rangle), \quad \epsilon = \pm 1, \tag{42}$$

which are separable if $|\psi\rangle \in \mathcal{H}_\pm$.

5. Summary

The more conservation laws are present in a system (either classical or quantum) the better is our understanding of its behaviour and properties. In this paper, we have presented a method of designing a time evolution of a composite qubit–environment system which guarantees the conservation of a preselected qubit's observable (its energy for instance) in the absence of the related conservation law. This can be done by a proper choice of the initial state of the total system (dephasing state). We have also argued that such initial states are entangled in general, yet one can disentangle them in principle.

Besides obvious applications, it may be also possible to apply our results either for testing the precision of initial state preparation of qubit–boson systems or for examining whether a chosen Hamiltonian properly describes a quantum system. The conservation of $\langle\Lambda\rangle$ is granted if one successfully prepares the initial state $|\Psi\rangle$ from equation (4) (or equation (27)). On the other hand, the dynamics of $\langle\Lambda\rangle$ can serve as a 'first test', usually simpler than e.g. tomographic methods, for the quality of applied state engineering. Having prepared the right initial state, any departure from $\langle\Lambda\rangle = \mathrm{cst}$. could indicate either significant influence of noise (decoherence) or the necessity of modifying the system Hamiltonian (by including nonlinear terms, for instance).

To select the right initial state a solution of the Riccati equation is required. Although neither the form nor even the existence of such a solution can be taken for granted in general, some useful criteria of solvability, applicable to a broad range of relevant physical systems, can be found in the literature [29]. Nowadays, when powerful computers and accurate numerical methods are accessible, it is a secondary issue, to say the least, that we cannot solve this equation analytically.

A preparation of the observable-conserving states $|\Psi\rangle$ (or $|\Phi\rangle$) would require highly sophisticated quantum engineering. One could try to construct the state $|\chi\rangle = X|\psi\rangle$, then 'tensorize' it with $|-\rangle \otimes |\chi\rangle$ and finally superpose the result with the separable qubit–environment state $|+\rangle \otimes |\psi\rangle$. Clearly, the situation becomes simpler if the state which we try to prepare is separable, as it is when $|\psi\rangle$ is an eigenstate of X.

Our description of 'state preparation' is extremely naive and it does not even pretend to be an experimental suggestion, especially when the initial qubit–environment state is entangled. This is a very serious limitation, but our work is purely theoretical and any deeper analysis of experimental perspectives is essentially beyond its scope. Despite these difficulties, we hope that with the continuous development of quantum state engineering techniques the construction proposed in this work can become useful for applications.

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