Stochastic Bounds for Markov Chains and how to use them for performance evaluation

J.M. Fourneau

Laboratoire PRiSM, CNRS UMR 8144

Université de Versailles St-Quentin

[a] Joint works with Nihal Pekergin (Univ. Paris XII), Mouad Ben Mamoun (Mohammed V University, Rabat), Ana Bušić (INRIA Trec), Tugrul Dayar (Bilkent University), and Jean-Marc Vincent (LIG-CNRS-INRIA Mescal).

Motivation

- Solving very large Markov chains.
- Solving a set of chains (worst case analysis).
- Qualitative properties of models based on Markov chains.
- Proof of algorithms based on Markov chains.

Solving Large Chains

- The composition of submodels in interaction allows modeling of large and complex systems.
- A tensor representation of MC, either in discrete-time or continuous-time [30, 43]:

$$P = \sum_i \otimes_j M_i^j.$$

- Associated to several High Level Formalisms (Stochastic Process Algebra, Stochastic Automata Networks, Superposition of Stochastic Petri Nets, etc..).
- An efficient storage of large chains.

- But numerical analysis of chains in steady-state is still difficult [43].
- Compute performance indices R defined as reward functions on the steady-state distribution:

$$R = \sum_i r(i)\pi(i).$$

- In general the tensor representation is less efficient than the usual sparse matrix form for basic operations required for numerical analysis.

Bounding the Rewards

- Exact values of the performance indices are sometimes not necessary.
- It is often sufficient to satisfy the Quality of Service (QoS) requirements.
- Bounding some reward functions is sufficient.

Bounds

- Linear algebra problem ($\pi = \pi P$), polyhedral properties (Courtois and Semal [17, 18], Goyal, Muntz, Lui, Rubino and Buchholz [8]).
- Markov Decision Process (Van Dijk [49]).
- Stochastic Bounds (bounds of the sample-paths, coupling) (Stoyan [44, 45], Kijima [32], Shaked, Shantikumar[42]).
- Here : stochastic comparison and stochastic monotonicity based on linear algebra, not on sample-path theorem or coupling (stochastic arguments).

Methodology

- We have to model a problem using a very large Markov chain and compute its steady-state distribution.
- Design algorithmically a new chain (transition matrix) such that:
  - The reward functions will be upper or lower bounds of the exact reward functions.
  - The new matrix is simpler to solve (smaller or with an easy structure).
- Based on stochastic ordering and monotonicity of Markov chains, lumpability or censoring for building smaller chains) and patterns for the derivation of structured DTMC.

Motivation again: worst case analysis

- Models where some parameters are not perfectly known.
- For instance: transition probabilities are in some interval.
- Solving the worst case in the set of DTMC (i.e. the worst average reward).
- How to find the "worst" matrix in a set ?
- For steady-state and transient rewards, and absorption time or probabilities.
- Based on stochastic orderings for random variables and Markov chains, monotonicity of DTMC.

Motivation continued: Qualitative Properties

- Prove that a steady-state or transient reward or an absorbing time is increasing with a parameter or the DTMC.
- Prove the convergence of algorithms based on a Markov chain.
- Based on the monotonicity of the DTMC.

Classical techniques: Strong Stochastic Bounds

- Total ordering of the states.
- Strong stochastic ordering of the chain.
- Steady-state analysis.
- Restriction (here) : Discrete Time Markov Chains (DTMC) with **finite state space** $E = \{1, \ldots, n\}$ (n is the size of the chain) and **total order on the state space**.
- Continuous-Time MC : will be studied after uniformization
- $P_{i,*}$ will refer to row i of P.

Comparison of Random Variables

- The strong stochastic ordering is defined by the set of non-decreasing functions (Stoyan [44]).
- **Definition 1** *Let X and Y be random variables taking values on a totally ordered space. Then $X <_{st} Y$ if and only if $E[f(X)] \leq E[f(Y)]$ for all non decreasing functions f whenever the expectations exist.*

Discrete states

Definition 2 *If X and Y take values on the finite state space $\{1, 2, \ldots, n\}$ with p and q as probability distribution vectors, then $X <_{st} Y$ if and only if $\sum_{j=k}^{n} p_j \leq \sum_{j=k}^{n} q_j$ for $k = 1, 2, \ldots, n$.*

Example

$$(0.1, 0.3, 0.2, 0.1, 0.3) <_{st} (0, 0.4, 0, 0.3, 0.3)$$

because

$$\begin{cases} 0.3 & \leq \ 0.3 \\ 0.1 + 0.3 & \leq \ 0.3 + 0.3 \\ 0.2 + 0.1 + 0.3 & \leq \ 0 + 0.3 + 0.3 \\ 0.3 + 0.2 + 0.1 + 0.3 & \leq \ 0.4 + 0 + 0.3 + 0.3 \\ 0.1 + 0.3 + 0.2 + 0.1 + 0.3 & \leq \ 0 + 0.4 + 0 + 0.3 + 0.3 \end{cases}$$

Example

- $x = (0.1, 0.3, 0.2, 0.1, 0.3)$ and $y = (0, 0.5, 0, 0.2, 0.3)$ are not st-comparable because:
- $0.1 + 0.3 \leq 0.2 + 0.3$; thus $y <_{st} x$ is not true.
- $0.2 + 0.1 + 0.3 \geq 0 + 0.2 + 0.3$; thus $x <_{st} y$ is not true.

St-Bounds

- Average population, loss rates or tail probabilities are non decreasing functions.
- Bounds on the distribution imply bounds on these performance indices as well.
- St-bounds are valid for transient distributions as well as the steady state (we first study the steady-state here).

Comparison for Markov Chains

- Monotonicity [31] and comparability of the transition probability matrices yield sufficient conditions for the stochastic comparison of MC.
- **Definition 3 (st-Comparison of Stochastic Matrices)** *Let P and Q be two stochastic matrices. $P <_{st} Q$ if and only if $P_{i,*} <_{st} Q_{i,*}$ for all i.*

st-Monotone Matrix

- **Definition 4 (St-Monotone Matrix)** *Let P be a stochastic matrix, P is st-monotone if and only if for all u and v, if $u <_{st} v$ then $uP <_{st} vP$.*
- St-monotone matrices are completely characterized (this is not true for other orderings, see [5]).
- **Definition 5** *Let P be a stochastic matrix. P is st-monotone if and only if for all u and v, $u <_{st} v$ implies that $uP <_{st} vP$.*
- **Property 1** *Let P be a stochastic matrix, P is st-monotone if and only if for all i, $j > i$, we have $P_{i,*} <_{st} P_{j,*}$*

Examples

- $\begin{bmatrix} 0.1 & 0.2 & 0.6 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.6 \\ 0.0 & 0.1 & 0.3 & 0.6 \\ 0.0 & 0.0 & 0.1 & 0.9 \end{bmatrix}$ is monotone.
- $\begin{bmatrix} 0.1 & 0.2 & 0.6 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.6 \\ 0.0 & 0.1 & 0.3 & 0.6 \\ 0.1 & 0.0 & 0.1 & 0.8 \end{bmatrix}$ is not monotone.

Fundamental theorem

Theorem 1 *Let $X(t)$ and $Y(t)$ be two DTMC and P and Q be their respective stochastic matrices. If*

- $X(0) <_{st} Y(0)$,
- *st-monotonicity of at least one of the matrices holds,*
- *st-comparability of the matrices holds, that is, $P_{i,*} <_{st} Q_{i,*}$ $\forall i$.*

Then $X(t) <_{st} Y(t), t > 0$.

Relations and

- Thus, assuming that P is not monotone, we obtain a set of inequalities on the elements of Q:

$$\begin{cases} \sum_{k=j}^{n} P_{i,k} & \leq & \sum_{k=j}^{n} Q_{i,k} & \forall\ i,j \\ \sum_{k=j}^{n} Q_{i,k} & \leq & \sum_{k=j}^{n} Q_{i+1,k} & \forall\ i,j \end{cases} \quad (1)$$

- It is possible to use a set of equalities, instead of inequalities:

$$\begin{cases} \sum_{k=j}^{n} Q_{1,k} & \sum_{k=j}^{n} P_{1,k} & \\ \sum_{k=j}^{n} Q_{i+1,k} & max(\sum_{k=j}^{n} Q_{i,k}, \sum_{k=j}^{n} P_{i+1,k}) & \forall\ i,j \end{cases}$$

- Properly ordered (in increasing order for i and in decreasing order for j in previous system), a constructive way to obtain a stochastic bound (ALGORITHMS).

Vincent's Algorithm

Construction of an upper bound Q :$P <_{st} Q$ and Q is $<_{st}$ monotone

<u>*Column n:*</u>

$Q_{1,n}$ $P_{1,n}$;

For $i = 2$ **to** n **Do** $Q_{i,n} = \max(P_{i,n}, Q_{i-1,n})$;

<u>*Column j, $n-1 \geq j \geq 2$:*</u>

For j $n-1$ **downto** 2 **Do**

$Q_{1,j}$ $P_{1,j}$;

For i 2 **to** n **Do**

$Q_{i,j}$ $\max(\sum_{k=j}^{n} P_{i,k}, \sum_{k=j}^{n} Q_{i-1,k}) - \sum_{k=j+1}^{n} Q_{i,k}$;

End

End

<u>*Column 1:*</u>

For i 1 **to** n **Do** $Q_{i,1}$ $1 - \sum_{k=2}^{n} Q_{i,k}$;

An example

$$P1 \quad \begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.2 & 0.0 \\ 0.1 & 0.7 & 0.1 & 0.0 & 0.1 \\ 0.2 & 0.1 & 0.5 & 0.2 & 0.0 \\ 0.1 & 0.0 & 0.1 & 0.7 & 0.1 \\ 0.0 & 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$$

- Once an element is obtained, we can compute the element on the left and below.
- Begin with element $(1, n)$.
- Proceed by row or by column.
- The summations $\sum_{k=j}^{n} Q_{i-1,j}$ and $\sum_{k=j+1}^{n} Q_{i,j}$ are already computed when we need them. Store to avoid computations.

First steps

- First row is unchanged:

$$\begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.2 & 0.0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}.$$

First column

- Compute column n (st-monotonicity implies that the elements are non decreasing):

$$\begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.2 & 0.0 \\ & & & & 0.1 \\ & & & & 0.1 \\ & & & & 0.1 \\ & & & & 0.5 \end{bmatrix}.$$

Next Column

- Compute column $n-1$ (st-monotonicity implies that the sums of the last two elements in a row are non decreasing):

$$\begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.2 & 0.0 \\ & & & 0.1 & 0.1 \\ & & & 0.1 & 0.1 \\ & & & 0.7 & 0.1 \\ & & & 0.3 & 0.5 \end{bmatrix}.$$

- Finally $Q = v(P1) = \begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.2 & 0.0 \\ 0.1 & 0.6 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.0 & 0.1 & 0.7 & 0.1 \\ 0.0 & 0.1 & 0.1 & 0.3 & 0.5 \end{bmatrix}$.
- $\pi_{P1} = (0.180, 0.252, 0.184, 0.278, 0.106)$.
- $\pi_Q = (0.143, 0.190, 0.167, 0.357, 0.143)$.
- We can check that: $\pi_{P1} <_{st} \pi_Q$.
- Expectation: 1.87 for $P1$ and 2.16 for $v(P1)$.

Irreducibility of Q

- **Definition 6** *We denote by $v(P)$ the matrix obtained after application of Vincent's Algorithm to a stochastic matrix P.*
- Due to the subtraction operations, some elements of $v(P)$ may be zero even if the corresponding elements in P are non zero.
- It may happen that matrix $v(P)$ computed by Vincent's algorithm is not irreducible, even if P is irreducible.
- If matrix $v(P)$ is reducible, it has one essential class of states. It is still possible to compute the steady-state distribution for this class.

- New algorithm (IMSUB) which does not delete transitions while computing the bound.
- **Theorem 2** *Let P be an irreducible finite stochastic matrix. Matrix Q computed from P with IMSUB is irreducible if and only if*
  - $P(1,1) > 0$,
  - *every row of the lower triangle of matrix P contains at least one positive element.*

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.2 & 0.0 \\ 0.1 & 0.7 & 0.1 & 0.0 & 0.1 \\ 0.2 & 0.1 & 0.5 & 0.2 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.3 \\ 0.0 & 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} \quad Q = \begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.2 & 0.0 \\ 0.1 & 0.6 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.5 & 0.1 & 0.1 \\ \boxed{0.0} & \boxed{0.0} & \boxed{0.0} & 0.7 & 0.3 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \end{bmatrix}$$

- States 0, 1 and 2 are transient.

Optimality

- **Theorem 3 (Optimality)** *Vincent's algorithm provides the smallest st-monotone upper bound for a matrix P: i.e. if we consider U another st-monotone upper bounding DTMC for P then $v(P) <_{st} U$ [1].*
- Proof based on properties of (max,+) equations.
- However bounds on the probability distributions may still be improved.
- The former theorem only states that Vincent's algorithm provides the smallest matrix according to the st-ordering of matrices.
- The sparse matrix and tensor versions of most of the algorithms are straightforward.

Lower Bound

- Based on the same relations.
- Consider another ordering for the index of the rows and the columns.

$$\begin{array}{lcl} n & \rightarrow & 1 \\ n-1 & \rightarrow & 2 \\ & \cdots & \\ 1 & \rightarrow & n \end{array}$$

- Another operator (min instead of max).

Methodology for simplification

- $v(P)$ is, in general, as difficult as P to analyze.
- matrix $v(P)$ may have many more positive elements than matrix P and it may be even completely filled.
- Use the inequalities (degree of freedom) and build a matrix simpler to analyze.
- Easy to solve : matrices with structural or numerical properties (Pattern, Class C) or smaller matrices (lumpability, censored MC).
- Use ad-hoc algorithms for the numerical resolution of structured matrices or usual algorithms when the size of the bounding chain is small enough.
- No new assumptions on P.

Ordinary lumpability

- Used by Truffet with st-comparison to model ATM switch [48].
- Lumpability implies a state space reduction. (decomposition of the chain into macro-states)
- **Definition 7 (ordinary lumpability)** *Let X be an irreducible finite DTMC, Q its matrix, let A_k be a partition of the states. X is ordinary lumpable according to A_k, iff for all states e and f in the same arbitrary macro state A_i, we have:*

$$\sum_{j \in A_k} q_{e,j} = \sum_{j \in A_k} q_{f,j} \quad \forall \; macro-state \;\; A_k$$

- Ordinary lumpability constraints are consistent with st-monotonicity.
- An algorithm is proposed by Truffet [48].

Truffet's algorithm

- Assume that states are ordered according to the macro-state partition.
- Ordinary lumpability = constant row sum for the block
- The algorithm computes the matrix row by row with some particular work for block boundaries.
- Due to st-monotonicity, the maximal row sum is reached for the last row of the block.
- The values of the lumped matrix are obtained for the last row sum of a block (except for the last non zero block).

Example

- $P6 = \left[\begin{array}{cc|ccc} 0.5 & 0.2 & 0.2 & 0.0 & 0.1 \\ 0.2 & 0.4 & 0.2 & 0.2 & 0.0 \\ \hline 0.2 & 0.3 & 0.1 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0.3 & 0.3 & 0.3 & 0 & 0.1 \end{array}\right].$
- We divide the state-space into two macro-states: $(1, 2)$ and $(3, 4, 5)$.

- The bounding matrix and the row sums for the first block:

$$\left[\begin{array}{cc|ccc} 0.5 & 0.2 & 0.2 & 0.0 & 0.1 \\ 0.2 & 0.4 & 0.2 & 0.1 & 0.1 \\ \hline & & & & \\ & & & & \\ & & & & \end{array}\right] \qquad \begin{array}{c} 0.3 \\ 0.4 \\ \hline \\ \\ \\ \end{array}$$

- The lumpable matrix and the lumped one:

$$\left[\begin{array}{cc|ccc} 0.4 & 0.2 & 0.3 & 0.0 & 0.1 \\ 0.2 & 0.4 & 0.2 & 0.1 & 0.1 \\ \hline & & & & \\ & & & & \end{array}\right] \qquad \left[\begin{array}{cc} 0.6 & 0.4 \\ & \end{array}\right]$$

Various implementations

- LMSUB: Sparse matrix implementation of Truffet's algorithm [13].
- LIMSUB: add the irreducibility constraint (as IMSUB) [23].
- SAN2LMSUB: the input is a sum of tensor products. The output is a sparse matrix [26].

A problem in Optical switch dimensionning

- For Optical Packet Switching (ie not an OBS, not a circuit)
- Deflection routing
- Fixed Packet Size
- No buffer but some Fiber Delay Loops
- The ROM/ROMEO architecture proposed by Alcatel
- m add and drop links. 4 transit links.

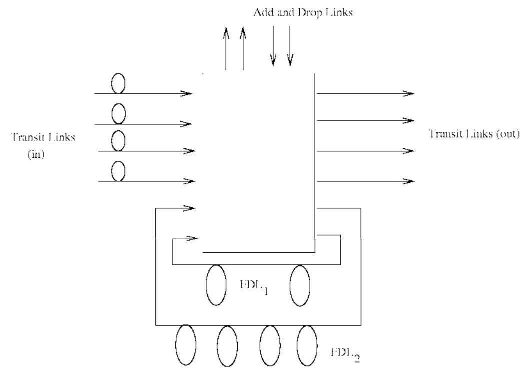

Figure 1: ROM core network architecture

Shortest Path Deflection Routing

- Switches attempt to forward packets along a shortest hop path to their destination.
- Each link can send a finite number of packets per time-slot (the link capacity).
- No Buffer: incoming packets have to be sent immediately
- If the number of packets which requires a link is larger than the link capacity, some of them will be misdirected or deflected
- Deflected packets will travel on longer paths.

Fiber Delay Loops and Local Deflection

- Adding FDL helps to reduce the effect of a deflection.
- If a packet must be deflected, we store it in the FDL instead of sending it a wrong direction.
- Using FDL is denoted as local deflection (much shorter delay than global deflection).
- The FDL is a fixed delay not a buffer. A packet leaves after a fixed delay. If there is an empty slot before, it cannot use it.
- Configuration an arbitrary number of FDL (theory) and with one or two FDL (numerical analysis)
- Delays: 1 or 2 (any integer for the theory).
- Question: How many FDL and size of the loops to obtain a sufficiently low number of global deflections ?

Assumtions and Model

- 4 input links, 4 output links, f wavelentghs per link
- iid batch arrivals
- uniform or almost uniform routing
- a small network (model of a core)
- a fixed probability d to leave the network
- Markov chain
- without FDL: a simple numerical computation
- 1 FDL and *delay* 1 a small Markov chain, usual algorithm on MC (GTH)
- Markov chain of order 2: to model the FDL with delay equal to 2 you must know the number of packets stored at time $t-1$ and $t-2$. Some possible reduction technique (lumpability) but it remains difficult.

Method

- Bounds on the number of global deflection rather than exact result
- Stochastic bounds are usually based on total ordering
- A lot of useless constraints with the total ordering
- Here with a convenient partial order, the initial model is monotone. AVOID to build a monotone BOUND.
- 3 Steps
  - Proof of the monotony of the initial model with n FDL
  - Deriving a bound: not monotone but smaller chain
  - Numerical analysis of the bound

Design of the bound

- Make the bound lumpable
- Do not lump states without packets in the longest FDL
- Upper bound: Change the transitions to mimic a state with a larger number of packets in the longest FDL
- Lower bound: Change the transitions to mimic a state with a smaller number of packets in the longest FDL
- Because of the fundamental result
- Check the accuracy using lower bound and upper bounds.

Numerical results

- Batch: Truncated Poisson, or All or Nothing
- very accurate results

Table 1: Truncated Poisson distribution, f 128, block size 16

2*rate	Mean real deflection	
	lower b.	upper b.
0.8	1.3634e-26	2.0339e-25
0.85	4.5848e-16	4.4175e-14
0.9	1.5349e-09	1.5737e-08
0.95	6.0196e-08	7.9197e-08
0.99	8.3536e-08	9.1247e-08

Censored Markov Chains

- Consider a DTMC with finite state space $S = E \cup E^c$, $E \cap E^c = \emptyset$.
- The censored DTMC with censoring set E watches the chain when it is in block E.
- For the steady-state, equivalent to the stochastic complement proposed by Meyer in [37].

  Consider a block decomposition of Q: $\begin{pmatrix} Q_E & Q_{EE^c} \\ Q_{E^cE} & Q_{E^c} \end{pmatrix}$.
- The stochastic complement matrix for block E: $S = Q_E + Q_{EE^c}(I - Q_{E^c})^{-1} Q_{E^cE}$.
- Solving $\pi_S = \pi_S S$ with $\sum \pi_S = 1$,
- π_S is the conditional steady-state probabilities for block E given that the DTMC is in block E: $\pi_S = \pi_E / \sum \pi_E$.

CMC and Bounds: Why

- Size: Q and Q_{E^c} are in general very large, so it is difficult to compute $(I - Q_{E^c})^{-1}$ ($I - Q_{E^c}$ is not singular if Q is not reducible [37]).
- Information: Q_E is known but the other blocks may be computed or not...
- Both cases: Deriving bounds on S.
- Avoid to build Q_{E^c} during the generation of the model. and compute $(I - Q_{E^c})^{-1}$?
- Construct $\overline{S}$ such that $S <_{st} \overline{S}$.
- Construct the monotone bound for $\overline{S}$ by Vincent's algorithm (R).
- $\overline{S} <_{st} R$ and R is $<_{st}$-monotone. Therefore: $\pi_S <_{st} \pi_R$

CMC and Bounds: Information

- Known: Q_E The simplest way [47] is to put the slack probability $\vec{\beta}$ to the **last** column for the **upper** bounding case, to the **first** column for the **lower** bounding case.
- Known: Q_E and $Q_{E^c,E}$: Better repartition of the slack probability : DPY algorithm [22], proved optimal, (compute the ST-Max of all rows of a normalized version of $Q_{E^c,E}$, left-multiply by $\vec{\beta}$ and add to Q_E)
- Known: Q_E, $Q_{E^c,E}$, and Q_{E,E^c}: BDF algorithm
- Known: Q_E, $Q_{E^c,E}$, Q_{E,E^c} and some transitions in Q_{E^c,E^c}: several algorithms
- Main idea: the more information you provide, the more accurate the bound.

Example

$$Q = \left[\begin{array}{ccc|cc} & \mathbf{Q}_E & & \mathbf{Q}_{EE^c} & \\ 0.1 & 0.2 & 0.4 & 0.2 & 0.1 \\ 0.3 & 0.1 & 0 & 0.4 & 0.2 \\ 0.1 & 0 & 0 & 0.6 & 0.3 \\ \hline 0.1 & 0.2 & 0 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.2 & 0.1 & 0.1 \\ & \mathbf{Q}_{E^cE} & & \mathbf{Q}_{E^c} & \end{array}\right]$$

$$S = \begin{bmatrix} 0.1831 & 0.3661 & 0.4508 \\ 0.4661 & 0.4322 & 0.1017 \\ 0.3492 & 0.4983 & 0.1525 \end{bmatrix} \qquad \vec{\beta} = \begin{bmatrix} 0.3 \\ 0.6 \\ 0.9 \end{bmatrix}$$

Q_E is known

Truffet's algorithm for the bound $\overline{S'}$ and Vincent's algorithm for the monotone bound R'

$$\overline{S'} \quad Q_E + \vec{\beta}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0.1000 & 0.2000 & 0.7000 \\ 0.3000 & 0.1000 & 0.6000 \\ 0.1000 & 0.0000 & 0.9000 \end{bmatrix}$$

$$R' = \begin{bmatrix} 0.1000 & 0.2000 & 0.7000 \\ 0.1000 & 0.2000 & 0.7000 \\ 0.1000 & 0.0000 & 0.9000 \end{bmatrix}$$

Q_E and $Q_{E^c,E}$ are known

-

$$\overline{S} = Q + \beta\begin{bmatrix} 0.25 & 0.5 & 0.25 \end{bmatrix} = \begin{bmatrix} 0.175 & 0.350 & 0.475 \\ 0.450 & 0.400 & 0.150 \\ 0.325 & 0.450 & 0.225 \end{bmatrix}$$

- Monotone and upper-bounding matrix of $\overline{S}$:

$$R = \begin{bmatrix} 0.1750 & 0.3500 & 0.4750 \\ 0.1750 & 0.3500 & 0.4750 \\ 0.1750 & 0.3500 & 0.4750 \end{bmatrix}$$

Q_E, Q_{E,E^c}, and $Q_{E^c,E}$ are known

- New algorithm (Busic,Djafri,Fourneau)
-

$$S_{BDF} = \begin{bmatrix} 0.18 & 0.36 & 0.46 \\ 0.46 & 0.42 & 0.12 \\ 0.34 & 0.48 & 0.18 \end{bmatrix}$$

- Remember that the exact result is:

$$S = \begin{bmatrix} 0.1831 & 0.3661 & 0.4508 \\ 0.4661 & 0.4322 & 0.1017 \\ 0.3492 & 0.4983 & 0.1525 \end{bmatrix}$$

Getting more

- Improving accuracy.
- Transient analysis of rewards.
- Absorbing DTMC.
- Qualitative properties.
- Worst Case Analysis.

Improving accuracy

- Apply some transformations [19] on P before Vincent's algorithm.
- First, $\alpha(P,\delta) = (1-\delta)Id + \delta P$, for $\delta \in (0,1)$.
- It has no effect on the steady-state distribution.
- It has a large influence on the effect of Vincent's algorithm.
- **Theorem 4** *Let P be a DTMC, and two different values $\delta_1, \delta_2 \in (0,1)$ such that $\delta 1 < \delta 2$, Then $\pi_{v(\alpha(P,\delta 1))} <_{st} \pi_{v(\alpha(P,\delta 2))} <_{st} \pi_{v(P)}$.*

A good value for δ

- **Definition 8** *A stochastic matrix is said to be row diagonally dominant (RDD) if all of its diagonal elements are greater than or equal to 0.5.*
- **Corollary 1** *Let P be a RDD DTMC, then $v(P)$ and $v(\alpha(P))$ have the same steady-state probability distribution.*
- Idea : For a RDD matrix, the diagonal serves as a barrier for the perturbation moving from the upper-triangular part to the strictly lower-triangular part $v(P)$.
- δ 1/2 is sufficient to make an arbitrary stochastic matrix RDD.
- Thus the transformation $P/2 + Id/2$ provides the best bound for these linear transformations.

Polynomials

- To obtain more accurate bounds.
- **Definition 9** *Let $\mathcal{D}$ be the set of polynomials $\Phi()$ such that $\Phi(1) = 1$, Φ different of Identity, and all the coefficients of Φ are non negative.*
- **Proposition 1** *Let $\Phi()$ be an arbitrary polynomial in $\mathcal{D}$, then $\Phi(P)$ has the same steady-state distribution than P.*
- **Theorem 5** *Let Φ be an arbitrary polynomial in $\mathcal{D}$, Algorithm 1 applied on $\Phi(P)$ provides a more accurate bound than the steady-state distribution of $v(P)$ i.e.:*

$$\pi_P <_{st} \pi_{v(\Phi(P))} <_{st} \pi_{v(P)}.$$

- But it is not always true that the higher the degree the more accurate the bounds...

Analysis of absorbing time

- **Theorem 6** *[3] Let X and Y two DTMC on state space $0..n$ absorbing in n (only one absorbing state), with stochastic matrices P and Q assume that:*
  1. *X_0 Y_0*
  2. *P or Q is st-monotone*
  3. *$P <_{st} Q$*

  then $T_Y <_{st} T_X$ where T_X is the absorbing time in n for chain X.
- The output of LMSUB may be a lumped matrix which is still absorbing (some technical conditions to check).
- It is much easier to compute the fundamental matrix on the lumped chain.

Qualitative Properties

- How to prove that an absorbing time (or a st-st reward) is increasing with a parameter of the model ?
- How to prove some algorithms based on Markov chains and mean interaction.
- A simple example rather than a general theory: End to end delay with SP deflection routing [11].
- Deflection routing: used when it is impossible to store packets waiting for the best output (typically all optical switch).
- Shortest Path Deflection routing: try shortest paths but use deflection when the number of packets exceeds the link capacity.

Effect of a deflection

- **Definition 10 (Symmetric Graph)** *A graph $G = (V, E)$ is symmetric iff for all i and j nodes in V, if (i, j) is a directed edge in E, (j, i) is also in E.*
- **Property 2** *In a symmetric graph, the deflected packet originally at distance k can jump at distance $k-1$ or $k+1$ or is still at distance k (because of the shortest-path deflection routing).*
- Let p (unknown) be the deflection probability and $R(p)$ the transition matrix.
- Major Assumption: Topology ∣ Independence of packets ∣ Uniform distribution for the O-D imply an aggregated Markov chain whose state is the distance to the destination.
- 0 is an absorbing state.

Topology and Initial Distribution

- An odd ring
- In the example, the size of the graph (sz) is 7.
- Thus the states of the chain are $0, 1, 2, 3$.
- Uniform destination and source (but source / destination).
- Two nodes at each distance.
- Initial distribution for the ring with 7 nodes: $(0, 1/3, 1/3, 1/3)$.

Transitions for an odd ring

- If k 0 stay in the same state.
- If the packet is not deflected: transition form k to $k-1$ with probability $1-p$.
- If the packet is deflected: transition from k to $k+1$ except when k sz where the packet is kept at distance sz after deflection (due to the odd ring topology).
-

$$R(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1-p & p \end{bmatrix}.$$

Properties

- The matrix is monotone for all value of p; this is always true for an odd ring and always false for an even ring.
- If $p1 > p2$ then $R(p2) <_{st} R(p1)$.
- $X(p)$: Absorption time in 0: end to end delay in the network (without taking into account the insertion delay at the interface).
- $E(X(p)) < \infty$ if $p < 1$.

Main Results

- If $p1 > p2$ $X(p1) <_{st} X(p2)$.
- $E(X(p))$ is increasing with p.
- If we are able to find bounds on p, we can derive bounds on $X(p)$.
- For instance $pmin \leq p \leq pmax$ implies than $E(X(pmin)) \leq E(X(p)) \leq E(X(pmax))$.

Fixed Point: deflection prob. p, load u

- Little's law: $E(N) \quad \lambda E(X(p))$ with λ accepted arrival rate.
- Link Utilization: $u \quad \frac{E(N)}{2sz}$ because a directed ring with sz nodes has $2sz$ directed edges.
- This gives an increasing function $u = f(p)$.
- Another model $p \quad g(u)$:
- g is increasing and $g(1) < 1$. Indeed a conflict between k packets give $k - 1$ deflection.
- Thus you have a fixed point system $u = f(p)$ and $p = g(u)$.

Proving the existence of a solution

- f and g are increasing.
- $g(1) < 1$.
- f and g are upper-bounded.
- **Theorem 7** *As the sequence* $(p_0 \quad 0, p_{i+1} \quad g(f(p_i)))$ *is increasing and upper-bounded, it has a limit which is a solution of the fixed point system.*

  A proof of existence and the way for an algorithm.

Worst Case Analysis

- For analysis of stochastic matrices which are not completely specified.
- For instance, the transition probabilities are not exactly known; we just give some intervals.
- $M = \begin{bmatrix} 0 & 1-a-b & b & a \\ 1-a/2 & a/2 & 0 & a/2 \\ 1-b/2 & 0 & b/2 & b/2 \\ 1-a-b & 0 & 0 & a+b \end{bmatrix}$

  with $1/3 \leq a \leq 1/2$ and $1/4 \leq b \leq 1/3$.
- For steady-state analysis see recent paper by Buchholz [8] based on polyhedral theory.

A stochastic approach

- Allows more general results.
- Transient and steady state analysis.
- Time to Failure (absorption).
- Based on stochastic ordering and monotonicity.
- We only consider here matrices where elements are in intervals (a different approach is used in the section on icx-ordering).

Partially defined DTMCs

- Consider a set of stochastic matrices $P \in \mathcal{P}(L, U)$.
- $L \leq_{el} P \leq_{el} U, \quad \forall P \in \mathcal{P}$.
- Construction of extreme stochastic matrices $\overline{P}$ and $\underline{P}$ by Truffet [47] such that $\underline{P} <_{st} P <_{st} \overline{P}, \quad \forall P \in \mathcal{P}$

Truffet's 2nd Algorithm

Construction of the extreme upper bound $\overline{P}$ for the set $\mathcal{P}(L, U)$

For $i = 1$ **to** n **Do**
$\Delta_i = 1 - \sum_{j=1}^{n} L_{i,j}$;
For $j = n$ **downto** 1 **Do**
$\delta = \min(\Delta_i, (U_{i,j} - L_{i,j}))$;
$\overline{P}_{i,j} = L_{i,j} + \delta; \quad \Delta_i = \Delta_i - \delta$;
End
End

- Lower Bound obtained by adding Δ from beginning by the first column
- If $U_{i,*} = L_{i,*} + \Delta_i \ \forall i$, it leads to complete in the last column for the upper bound and in the first column for the lower bound
- A similar algorithm presented by Haddad and Moreaux for substochastic matrices to improve the polyhedral approach [29].

Optimality

- Let $\overline{Q}$ and $\underline{Q}$ be monotone matrices obtained by Vincent's algorithm for input matrices $\overline{P}$ and $\underline{P}$.
- $\overline{Q}$ and $\underline{Q}$ are optimal monotone bounds for the set $\mathcal{P}(L, U)$:
If monotone stochastic matrices A, B exist such that
$A <_{st} P <_{st} B \quad \forall P \in \mathcal{P}(L, U)$
then $A <_{st} \underline{Q}$ and $\overline{Q} <_{st} B$
- Stochastic bounds on the transient and steady-state distributions for the set of matrices defined by $\mathcal{P}(L, U)$:
$\Pi_{\underline{Q}}(t) <_{st} \Pi_P(t) <_{st} \Pi_{\overline{Q}}(t) \quad \forall t, \ \forall P \in \mathcal{P}(L, U)$

Increasing Convex Ordering

- A variability ordering.
- More complex than the usual st ordering.
- More accurate than st ordering when one deals with random variables.
- If $X <_{st} Y$ and $E(X) = E(Y)$ then X and Y are identically distributed.
- It is possible to consider the set of random variables with the same expectation and find the maximal or minimal r.v. according to the icx ordering.

Increasing Convex Ordering

- **Definition 11** *Let X and Y be two random variables taking values on a totally ordered space space. Then we say that X is smaller than Y in the increasing convex sense (icx),*

$$X <_{icx} Y \quad \textit{if} \quad E(f(X)) \leq E(f(Y))$$

for all increasing and convex functions f whenever the expectations exist.

- Thus "st" ordering (defined by increasing functions) implies "icx" ordering (defined by increasing and convex).

On discrete state space

$$X <_{icx} Y \iff \sum_{k=i}^{n}(k-i+1)\, x_k \leq \sum_{k=i}^{n}(k-i+1)\, y_k, \quad \forall i$$

$$\iff \begin{cases} x_n & \leq & y_n \\ x_{n-1} + 2x_n & \leq & y_{n-1} + 2y_n \\ x_{n-2} + 2x_{n-1} + 3x_n & \leq & y_{n-2} + 2y_{n-1} + 3y_n \\ & \dots & \\ x_1 + 2x_2 + \ldots + nx_n & \leq & y_1 + 2y_2 + \ldots + ny_n \end{cases}$$

Example

- Three probability vectors: $x = (0.5, 0.1, 0.1, 0.3)$, $y = (0.3, 0.2, 0.2, 0.3)$, and $z = (0.3, 0.2, 0.4, 0.1)$
- $x <_{icx} y$ as
  - $0.3 \leq 0.3$ and $0.1 + 2*0.3 \leq 0.2 + 2*0.3$
  - $0.1 + 2*0.1 + 3*0.3 \leq 0.2 + 2*0.2 + 3*0.3$
- The vectors x and z are not icx-comparable as
  - $x_3 = 0.3 > 0.1 = z_3$, but
  - $x_1 + 2x_2 + 3x_3 = 1.2 < 1.3 = z_1 + 2z_2 + 3z_3$.

icx-monotone DTMC

- Much harder constraints.
- Ben Mamoun's characterization for finite DTMC:
  P is icx-monotone iff $Z_{icx} P K_{icx} \geq 0$ component-wise with:

$$Z_{icx} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -2 & 1 \end{bmatrix} \quad K_{icx} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n-1 & n-2 & \dots & 1 \end{bmatrix}$$

No Optimal Bound for icx ordering of DTMC

- Consider $P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$,
- and $U1$ and $U2$ which are icx monotone upper bound of P:

$$U1 = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \quad U2 = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

- It is not possible to prove an optimal bound Q such that $P <_{icx} Q$, $Q <_{icx} U1$ and $Q <_{icx} U2$.
- Indeed the last column of Q must be $(0.1, 0.4, 0.5)^t$ which is not convex.

A Batch/D/1/N queue

- Buffer size for optical packet switch with constant packet size
- Without electronic conversion (no electronic buffer) : use Fiber Delay Loops instead
- Without wavelength conversion: 1 server per wavelength.
- K input links.
- ROM and ROMEO architectures (Alcatel)
- Batch/D/1/N queue
- We know the average arrival rate (easy to measure) and the maximal batch size K.
- Can we dimension the buffer ?

Steps of the analysis

- Note that the model is almost-icx monotone.
- Use icx-ordering.
- Find the worst arrival process according to icx-ordering and derive the Markov chain of the queue.
- Scale the chain to allow icx-comparison.
- Make the scaled Markov chain icx monotone.

Worst Case Arrival

- $A = (a_0, \dots, a_K)$ distribution of batch arrivals.
- $\alpha = E(A)$ is known.
- We assume: $N > K$ (engineering) and $\alpha < 1$ (stability).
- $\mathcal{F}_\alpha$ = the family of all distributions on the space $\{0, \cdots, N\}$ having expectation α
- icx-worst case distribution: $q = (\frac{N-\alpha}{N}, 0, \dots, 0, \frac{\alpha}{N})$:
- **Property 3 (Maximal R.V. (see Shantikumar))**

$$q \in \mathcal{F}_\alpha \quad \textit{and} \quad p \preceq_{icx} q, \ \forall p \in \mathcal{F}_\alpha$$

Matrix of the Chain

- $$\boldsymbol{P} = \begin{pmatrix} a_0 & a_1 & \cdots & a_K & 0 & \cdots & 0 \\ a_0 & a_1 & \cdots & a_K & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_K & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & a_0 & a_1 & \cdots & a_K \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & a_0 & \sum_{i=1}^{K} a_i \end{pmatrix}$$

- A bound of the arrival rate is not sufficient.
- The matrix must be monotone (and $\boldsymbol{P}$ is not...).

4 Steps

1. Build an upper icx-bound $\boldsymbol{Q}$ for each row using the worst arrival process. $\boldsymbol{Q}$ is not icx-monotone
2. Modify matrix $\boldsymbol{Q}$: $t_\delta(\boldsymbol{Q}) = \delta\boldsymbol{Q} + (1-\delta)\boldsymbol{Id}$
   t_δ: same steady-state distribution, move some probability mass to the diagonal elements to allow step 4.
3. Make the last row of $t_\delta(\boldsymbol{Q})$ increasing and convex
4. Change diagonal and sub-diagonal elements to make final matrix $\boldsymbol{B}$ icx-monotone

Main result

Theorem 8 *Suppose that*

$$\delta \leq \frac{1}{1 + \alpha U}, \tag{2}$$

where $U = \max_{r=2\dots K-1} \frac{r(K-r+1)}{K}$. *Then,*

1. $\boldsymbol{B}$ *is a stochastic matrix.*
2. $\boldsymbol{B}$ *is irreducible.*
3. $\boldsymbol{Q} <_{icx} \boldsymbol{B}$.
4. $\boldsymbol{B}$ *is icx-monotone.*

Accuracy: a numerical example

- The perturbation added by the monotonicity constraint is relatively small (i.e. difference between st-st distribution of $\boldsymbol{Q}$ and $\boldsymbol{B}$).
- The main error comes from the main assumption (we ONLY know the average and the max batch size).
- A state dependent batch.
- Back-pressure mechanism. When the queue size is large, a signal is sent to the sources of traffic to avoid congestion and shape the traffic.
- Shaping: same average (not that important, we can reduce) and smaller variability.
- Smaller variability: smaller K.
- Threshold: 80% of the buffer size.

Average number of packets in the queue

3*α	K−10			K−100		
	$\boldsymbol{S}$	$\boldsymbol{B}$	rel. error	$\boldsymbol{S}$	$\boldsymbol{B}$	rel. error
0.5	5.000e+00	5.000e+00	$< 10^{15}$	5.00e+01	5.00e+01	2.7e-05
0.8	1.880e+01	1.880e+01	$< 10^{15}$	1.93e+02	1.97e+02	1.5e-02
0.9	4.140e+01	4.140e+01	8.9e-09	3.69e+02	3.92e+02	6.3e-02
0.95	8.644e+01	8.645e−01	9.1e-05	5.45e+02	6.06e−02	1.1e-01
0.99	3.780e+02	3.984e−02	5.3e-02	7.95e+02	9.00e−02	1.3e-01

Table 2: Comparison of the mean queue length at the steady-state between the state dependent ($\boldsymbol{S}$) and the monotone upper bound ($\boldsymbol{B}$) for N 1000, $K = 10$ and $K = 100$.

Quantitative and Qualitative Results

- Performance Evaluation
- Reliability (MTTF, point availability)
- Model Checking [7, 25, 41] (but the answer may be "With the bound I am not able to answer True or False") and some operators have to be studied more carefully.
- Some performance indices are increasing functions of the parameters.
- Proof of the convergence of a method based on the iterative solution of subproblems if one of the subproblems is the analysis of a Markov chain.
- Is it possible to prove some well known approximate iterative methods in performance evaluation?

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