On the Generalisation of the Zipf-Mandelbrot Distribution and its Application to the Study of Queues with Heavy Tails

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Outline

- Motivation: Info. Theory, Statistical Mechanics & Quantification Theory ➔ Queues with bursty & heavy tails
- Maximum Entropy (ME) Formalism
  - ME and GB Solutions for the State Probability Distributions of queues with bursty (GE-type)tails
- An Extended ME (EME) Formalism
  - EME Solutions for the State Probability Distributions of queues with heavy tails
- Numerical Experiments
- Conclusions and further remarks on the ME extension to the analysis of open QNMs

Motivation: Information Theory, Statistical Mechanics & Quantification Theory → Queues with Bursty & Heavy Tails

To consider alternative analytic methodologies for queues with bursty and heavy tails, based on a balanced trade-off between simplified assumptions to reduce complexity and actual real life system behaviour, leading to credible and cost-effective approximations for performance prediction and optimisation of telecommunication systems.

Extended ME Formalism, Statistical Mechanics & Long-Range Interactions

In Statistical Mechanics:

Energy are assumed to be

- "Extensive" variables

such as total energy → ~ system size
(c.f., due to short-range interactions e.g., chemical bonds)
Similarly, entropy is also assumed to be extensive.

- "Non-extensive" variables

→energy no longer ~ system size

(c.f., due to long-range interactions such as gravity)

This makes life difficult in Statistical Mechanics!

Extended ME Formalism, Statistical Mechanics & Long-Range Interactions

- *Maximum Entropy (ME) Principle*

{max **Gibbs** 'Extensive' Entropy Function,

subject to a mean value constraint of a quantity
(e.g., system energy, # of molecules, volume)}

Applying Method of Lagrange Undetermined Multipliers

➔ **Geometric** Steady State Prob. Distribution

(Lagrange multipliers are
"intensive" variables ⇔ "extensive" ones with constrained means
(e.g., energy ⇔ temperature, volume ⇔ pressure,
# of molecules ⇔chemical potential etc)

Extended ME Formalism, Statistical Mechanics & Long-Range Interactions

- *Generalised Maximum Entropy Principle*

{max the **Havrda-Charvat** 'non-extensive' **entropy function** (a quantitative measure of classification, subject to a mean value constraint}

➔ ***Zipf-Mandelbrot*** *Steady* State Prob. *Distribution* with power-law (heavy) tails and non-extensivity real-valued parameter q

→ *Analogies with Statistical Mechanics applications* [Tsallis 1988] and *the analysis of queues with bursty traffic & heavy tails* [Assi 2000], [Kouvatsos & Assi 2002] / *LRD traffic & heavy tails* [Karmeshu & Sharma 2005], [Kouvatsos & Assi 2007]

The Zipf-Mandelbrot Distribution

The **Zipf-Mandelbrot** distribution is a discrete probability distribution. It is a power-law distribution on ranked data.

The probability mass function (pmf) is of the form

$$p(n,u,s) = \frac{(n+u)^{-s}}{\sum_{n=1}^{N}(n+u)^{-s}}$$

N - the number of elements

n, u - real numbers

s - the value of the exponent characterizing the distribution

The Zipf-Mandelbrot Distribution

- In the limit as N→∞, the sum $\sum_{n=1}^{N}(n+u)^{-s}$

  becomes the **Hurwitz-Zeta** function $\zeta(u,s)$

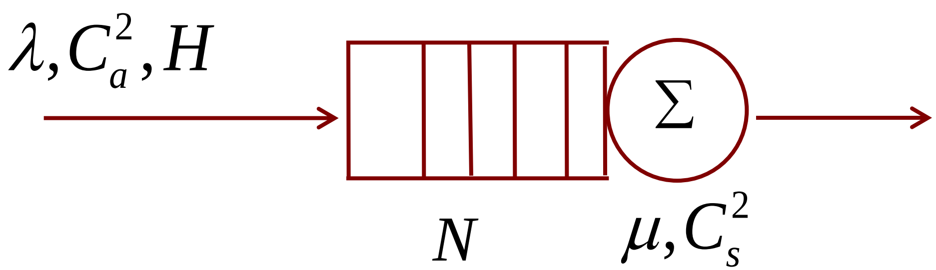
- For finite *N* and *u=0*, the **Zipf-Mandelbrot** law becomes **Zipf's** law (both commonly used in linguistics, Information Sciences, insurance, the modelling of events and ensemble theory in statistical mechanics)

- For infinite *N* and *u=0, the sum* is recognized as the **Zeta** distribution

The G/G/1 Queue & G/G/1/N Censored Queue with Bursty and/or LRD Traffic Flows

- A stable *G /G /1* Queue

- A censored *G /G /1 / N* Queue



{λ, C^2a}: the mean arrival rate and the interarrival sq. coef. of variation

H: Hurts parameter of the arrival process, *N*: Finite buffer capacity

{μ, C^2s}: mean service rate and sq. coef. of variation.

Maximum Entropy (ME) Formalism (Jaynes 1956a,b)

- *System Specification*
- *Optimisation Problem Formulation*
- *Analytic Methodology*
- *ME Solution*
- *Basic Relations*
- *Overview of ME and Queueing Network Models (QNMs)*

System Specification

- **Q**, General System;
- **$S = \{S_0, S_1, \ldots\ldots, S_n, \ldots\}$**

  Finite or countable infinite set of states;
- **$P(S_n)$**, state prob. distr. that **Q** is at state **S_n**;
- **$\{\langle f_k \rangle\}$, $k=1, 2, \ldots., m < |Q|$**,

  Set of prescribed mean values defined on the set of suitable functions:

  $\{f_1(Sn), f_2(Sn), \ldots., f_m(Sn)\}$

Optimisation Problem Formulation

$$\max_{P} \left\{ H(P) = \sum_{S_n \in S} P(S_n) \log P(S_n) \right\}$$

subject to

$$\sum_{S_n \in S} P(S_n) = 1,$$

$$\sum_{S_n \in S} f_k(S_n) P(S_n) = \langle f_k \rangle, \quad k = 1, 2, \ldots, m$$

where m is less than he number of possible states.

Apply the Method of
Lagrange's Undetermined Multipliers

ME Solution [Jaynes 1957a and 1957b]

$$P(S_n) = \frac{1}{Z}\prod_{k=1}^{m} x_k^{f_k(S_n)},$$

$$Z = e^{\beta_0} = \sum_{S_n \in S}\prod_{k=1}^{m} x_k^{f_k(S_n)},$$

Normalising Constant

$$x_k = e^{-\beta_k}, \quad k = 1, 2, \ldots, m$$

$\{\beta_k\}$ are the Lagrangian coefficients corresponding to constraints $\{\langle f_k \rangle\}$, k=1, 2, …, m

Basic Relations

- $$\frac{\partial \beta_0}{\partial \beta_k} = \langle f_k \rangle, \quad k = 1, 2, \ldots, m$$

- $$\max_P \left\{ H(P) \right\} = \beta_0 + \sum_{k=1}^{m} \beta_k \langle f_k \rangle$$

ME & EME FORMALISMS FOR ANALYSING OPEN (QNMs)

OPEN QNM WITH JOINT STATE PROBABILITY

$\{P(\underline{n}),\ \underline{n} = (n_1, n_2, \ldots, n_M),\ n_i \geq 0,\ n_i\ \ 1,2, \ldots, M\}$

⇩

CLASSICAL QUEUEING THEORY

MAX ENTROPY / EXTENTED ENTROPY

$Max_p\ H(p)$

⇔

MARGINAL MEAN VALUE CONSTRAINTS

ME FORMALISM & OPEN QNMs

⇩

PRODUCT FORM APPROXIMATION OF AN OPEN QNM

$$P(n) = \prod_{i=1}^{M} P_i(n_i)$$

⇩

INTERPRETATION

⇩

TRAFFIC FLOW ANALYSIS & ME QUEUE-BY-QUEUE DECOMPOSITION OF OPEN QNMs

A Stable G/G/1 Queue [Kouvatsos1994]

- Maximise Shannon's Entropy Functional

$$\max_{P}\left\{ H(P) = -\sum_{n=0}^{\infty} P(n)\log P(n) \right\}$$

subject to

- Normalisation, $\sum_{n=0}^{\infty} P(n) = 1,$
- Mean queue length, $\sum_{n=0}^{\infty} np(n) = L$
- Utilisation, $\sum_{n=0}^{\infty} h(n)p(n) = 1 - p(0) = \rho,\qquad \rho = \frac{\lambda}{\mu},\ \ 0 < \rho < 1$

$h(n) = 1$, if $n = 0$ or 0 otherwise

- Apply the Method of Lagrange's Undetermined Multipliers

A Stable G/G/1 Queue (Cont.)

- A ME **Generalised Geometric** Solution

$$P(n) = \begin{cases} 1-\rho, & n = 0 \\ (1-\rho)gx^{n}, & n \geq 1 \end{cases}$$

$$g = \frac{\rho^{2}}{(L-\rho)(1-\rho)}, \qquad x = \frac{L-\rho}{L}$$

where

$$L = \frac{\rho}{2}\left(1 + \frac{1+\rho C_s^2}{1-\rho}\right)$$

Mean queue length in M/G/1 queue (Pollaczek-Khintchine Formula) C^2_s: SCV of the service time, $C^2_a = 1$.

A Censored G/G/1/N Queue [Kouvatsos 1994]

- Maximise Shannon's Entropy Functional

$$\max_{P}\left\{ H(P) = -\sum_{n=0}^{\infty} P_N(n)\ \log P_N(n) \right\}$$

subject to

- The normalization, $\sum_{n=0}^{N} p_N(n) = 1$
- The mean queue length, $\sum_{n=0}^{N} n p_N(n) = L_N$
- The utilisation, $\sum_{n=0}^{N} h(n) p(n) = 1 - p_N(0) = U$
- Full buffer state probability, $\sum_{n=0}^{N} s(n) p_N(n) = \varphi = p_N(n),\ 0 < \varphi < 1$

where $h(n) = 1$, if $n = 0$ or 0 otherwise and $s(n)=1$, if $n=N$ or 0 otherwise

A Censored G/G/1/N Queue

- Apply the Method of Lagrange's Undetermined Multipliers
- Obtain a **Truncated** Generalised Geometric ME Solution (expressed in terms of the single step recursions) →

$$P_N(1) = gxP_N(0)$$
$$P_N(n) = xP_N(n-1) \quad n = 2, \; ...., \; N - 1$$
$$P_N(N) = yxP_N(N-1)$$

where

$$g = \frac{\sigma\rho\tau}{\sigma\rho + \tau(1-\sigma\rho)}, \quad x = \frac{\sigma\rho + \tau(1-\sigma)}{\sigma\rho + \tau(1-\sigma\rho)}, \quad y = \frac{\sigma\rho + \tau(1-\sigma\rho)}{\sigma + \tau(1-\sigma)}\frac{1}{\tau},$$

$$\rho = \frac{\lambda}{\mu}, \quad \sigma = \frac{2}{C_a^2 + 1} \quad \text{and} \quad \tau = \frac{2}{C_s^2 + 1}$$

The ME solution satisfies the flow balance condition $\lambda(1-\pi) = \mu(1-P_N(0))$, where π is the blocking probability.

Connection with the GE-type Distribution

- **Theorem:** The ME M/G/1 solution is equivalent to the queue length distr. of a stable M/G/1 queue with a *GE-type* service time prob. density function of the form

$$f(t) = (1-r)u_0(t) + r^2 \mu e^{-r\mu t}, \quad t \geq 0,$$

where $r = \frac{2}{C_s^2 + 1}$, $u_0(t) = +\infty$, if $t = 0$ or, 0, if $t \neq 0$.

Unit impulse function

This theoretical result can be shown by substituting **g**, **x** and **L** into the ME solution and equating its z-transform with the Laplace-Stieltjes transform of the service time [Kouvatsos1994].

Connection with the GE-type Distribution

- **Proof:** The Pollaczek-Khintchin z-transform of is

$$Q(z) = \frac{F^*(\lambda - \lambda z)(1-\rho)(1-z)}{F^*(\lambda - \lambda z) - z}$$

where F*(θ) is the Laplace-Stieltjes transform of the service time. This transform can be determined directly using the relation

$$Q(z) = \sum_{n=0}^{\infty} P(n) z^n, |z| \le 1$$

- This implies that

$$Q(z) = \frac{(1-\rho)\left[1 - xz(1-r)\right]}{(1-xz)}$$

Connection with the GE-type Distribution

- It can be easily verified that Q(0) = 1-ρ and Q(1) = 1. Equating the right-hand sides of both equation, substituting for x and ρ and solving for F*(λ-λz), the following result is obtained (with $r = 2 / (1+C_s^2)$)

$$F^*(\lambda - \lambda z) = \frac{r\mu + (1-r)(\lambda - \lambda z)}{r\mu + \lambda - \lambda z}$$

- Substituting θ for (λ- λz), becomes

$$F^*(\theta) = \frac{r\mu + (1-r)\theta}{r\mu + \theta} = (1-r) + \frac{r^2\mu}{r\mu + \theta}$$

- By inverting F*(θ), the result follows. Q.E.D

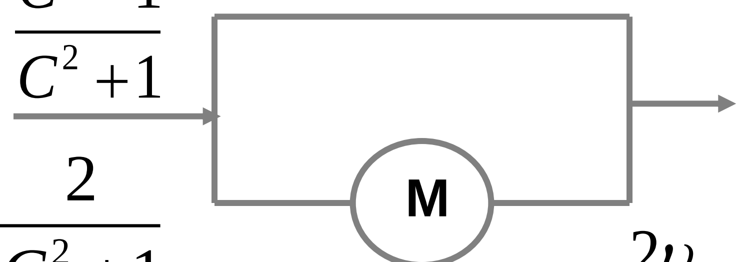
The GE-type Distribution

- The ME solution of a stable M/G/1 queue is exact if G ≡ GE. Similarly for a stable GE/G/1 queue
- The GE-type distr. with parameters α and β (0≤α≤ 1):

$$F(t) = 1 - \alpha e^{-\beta t}, \quad t \geq 0,$$

$$1 - \alpha = \frac{C^2 - 1}{C^2 + 1}$$

$$\alpha = \frac{2}{C^2 + 1}$$



$$\beta = \frac{2\upsilon}{C^2 + 1}$$

- The underlying counting process of the GE-type distr. is a compound Poisson process with Geo distributed batch sizes and mean batch size 1/ α = (C^2 + 1)/2.

Interpretation of GE-type distribution

- GE is an extremal case of the family of two-phase exponential distributions having the same {v, $C^2 (>1)$ }

- GE is a bulk type distribution with an underlying counting process equivalent to a Compound Poisson Process (CPP) with parameter $2v/(C^2+1)$ and a geometrically distributed bulk size with mean $(1+C^2)/2$ and SCV $(C^2-1)/(C^2+1)$ i.e.,

$$P(N_{cp}=n)=\begin{cases}\sum_{i=1}^{n}\frac{\sigma^i}{i!}e^{-\sigma}\binom{n-1}{i-1}\tau^i(1-\tau)^{n-i}, n\geq 1\\ e^{-\sigma}, \mathrm{n}=0\end{cases}$$

where N_{cp} is the random variable of the number of events per unit time.

Global Balance Solution for the Censored GE/GE/1/N Queue

- *GE-Type Algebra*

- GE-type Transition Rates
- GE-type Global Balance (GB) Equations
- GE-type GB Solution for the State Probability Distribution
- GE-Type GB Connection with ME Formalism
- GE-type Blocking Probability

Global Balance (GB) Solution for the Censored $GE_1/GE_2/1/N$ Queue

Let $GE_1 \sim GE(\sigma,\sigma\lambda)$ & $GE_2 \sim GE(\tau,\tau\mu)$, where

- σ, τ are the stage selection probabilities of the non-zero exponential branches of the GE_1 and GE_2, respectively
- $\sigma\lambda$, $\tau\mu$ are the arrival and service rates (at non-zero exponential branch of the queue), respectively i.e.,

$$\sigma = 2/(1+C^2a)$$

$$\tau = 2/(1+C^2s)$$

Global Balance (GB) Solution for the Censored GE1/GE2/1/N Queue (Cont.)

- The analysis utilises the bulk interpretation of the *GE-type distribution.*
- Suppose the number in the queue is $1 \leq k \leq N-1$ *when a bulk of size* $n \geq N-k$ arrives ➔ Implications
  - Then $N-k$ units are chosen randomly from the bulk to fill the empty spaces of the waiting room.
  - The remaining units of the bulk are considered to be lost.

GE-type GB Equations

$$\sigma\lambda\frac{\tau}{\tau(1-\sigma)+\sigma}P_0 = \tau\mu\sum_{k-1}^{N}(1-\tau)^{k-1}P_k$$

$1 \le i \le N-1$

$$(\sigma\lambda+\tau\ \mu)P_i = \sigma\lambda\frac{\tau\ \sigma(1-\sigma)^{i-1}}{\tau(1-\sigma)+\sigma}P_0+\sigma\lambda\left(\sum_{k=1}^{i-1}\sigma(1-\sigma)^{i-k-1}\right)P_k+\tau\ \mu\ \left(\sum_{k=i+1}^{N}\tau(1-\tau)^{k-i-1}\right)P_k$$

$$\tau\mu P_N = \sigma\lambda\frac{\tau(1-\sigma)^{N-1}}{\tau(1-\sigma)+\sigma}P_0+\sigma\lambda\left(\sum_{k=1}^{N-1}(1-\sigma)^{N-k-1}\right)P_k$$

The GE-type GB State Probability Distribution

$$P_N = P_0 \frac{\sigma\rho}{\tau(1-\sigma)+\sigma}\left(\frac{\sigma\rho+\tau(1-\sigma)}{\sigma\rho+\tau(1-\sigma\rho)}\right)^{N-1}, \qquad \rho = \frac{\lambda}{\mu}$$

$$P_k = P_0 \frac{\sigma\rho\tau}{\sigma\rho+\tau(1-\sigma\rho)}\left(\frac{\sigma\rho+\tau(1-\sigma)}{\sigma\rho+\tau(1-\sigma\rho)}\right)^{k-1} \qquad ,1 \le k \le N-1$$

$$P_0 = \frac{1-\rho}{1-\rho\frac{\sigma\rho+\tau(1-\sigma\rho)}{\sigma+\tau(1-\sigma)}\left(\frac{\sigma\rho+\tau(1-\sigma)}{\sigma\rho+\tau(1-\sigma\rho)}\right)^{N}}$$

GB Connection with the ME Formalism

- Maximise Entropy Functional

$$\max_{P}\left\{ H(P) = -\sum_{n=0}^{\infty} P_N(k)\ \log P_N(k) \right\}$$

subject to normalisation, utilisation, mean queue length and full buffer state probability constraints satisfying the flow balance Condition: λ(1-π) = μ(1-PN(0)), where π is the blocking probability.

- ME Solution

$$P_N(1) = gxP_N(0)$$

$$P_N(k) = xP_N(k-1) \quad k = 2, \ldots, N-1$$

where

$$P_N(N) = yxP_N(N-1)$$

$$g = \frac{\sigma\rho\tau}{\sigma\rho + \tau(1-\sigma\rho)}, \quad x = \frac{\sigma\rho + \tau(1-\sigma)}{\sigma\rho + \tau(1-\sigma\rho)}, \quad y = \frac{\sigma\rho + \tau(1-\sigma\rho)}{\sigma + \tau(1-\sigma)}\frac{1}{\tau},$$

$$\rho = \frac{\lambda}{\mu}, \qquad \sigma = \frac{2}{C_a^2 + 1} \qquad \text{and} \qquad \tau = \frac{2}{C_s^2 + 1}$$

The GE-type Blocking Probability

- The probability of an arrival to find the queue full, π, is given by

$$\pi = P_N(N) + \sum_{k=1}^{N-1} P_N(k)(1-\sigma)^{N-k} + P_N(0)\frac{\tau(1-\sigma)^N}{\tau(1-\sigma)+\sigma}$$

$$= \sum_{k=0}^{N} \delta(k) P_N(k)(1-\sigma)^{N-n}$$

where $\delta(k) = \begin{cases} \dfrac{\tau}{\tau(1-\sigma)+\sigma} & \text{k=0} \\ 1 & \text{k} \neq 0 \end{cases}$,

- The proof is based on the bulk interpretation of the compound Poisson arrival process to the queue and the GE-type service time distribution.

The GE-type Blocking Probability (cont.)

- The bulk finds N jobs in the GE/GE/1/N queue;

  Bulks arrive according to a Poisson ($\sigma\lambda$) process. Thus a tagged arriver will find N in the system with probability $P_N(N)$ (i.e., the same with that of a random observer).

- The bulk finds k jobs in GE/GE/1/N the queue ***(**1≤ k ≤N-1);*

  The size of the bulk is at least m = N-k+1 and the tagged arriver is one of those bulk members that will be blocked (turned away). Thus,

$$\sum_{k=1}^{N-1} P_N(k)\left(\sum_{m=N-k+1}^{\infty} m\frac{\sigma(1-\sigma)^{m-1}}{1/\sigma}\right)\frac{m-(N-k)}{m} = \sum_{k=1}^{N-1} P_N(k)(1-\sigma)^{N-k}$$

The GE-type Blocking Probability (cont.)

- The bulk finds 0 jobs the GE/GE/1/N queue;

The bulk size m is at least (N+1), at most m-(N+1) jobs choose the null GE branch from the front part of the bulk and the tagged arriver is one of those bulk units that will be blocked (turned away). Thus,

$$P_N(0)\sum_{m=N+1}^{\infty}\frac{m\sigma(1-\sigma)^{m-1}}{1/\sigma}\sum_{k=0}^{m-(N+1)}\tau(1-\tau)^k\frac{m-N-k}{m}=P_N(0)\frac{\tau(1-\sigma)^N}{\tau(1-\sigma)+\sigma}$$

- The form of the GE-type blocking probability, π , of the GE/GE/1/N queue is obtained by adding the probabilities of these three mutually exclusive events.

Havrda-Charvat Generalised Entropy Function

- The Havrda-Charvat generalised parametric entropy function, *Sq*, is defined by [Havrda & Charvat 1967]

$$S_q = \frac{C\left(1 - \sum_{i=0}^{\infty} p_i^{\,q}\right)}{q-1}$$

p_i, $i=0,1,\ldots$ are the state probabilities of the queue;

q is a real number measuring the degree of non-extensivity of the queue ;

C is a positive constant ;

Sq is a generalised measure of uncertainty in dynamic systems, which reduces to Shannon entropy function at the non-extensivity parameter $q \to 1$ limit H.

Generalised Entropy Maximisation: Generalization of Boltzmann Gibbs Statistics

- In Statistical Mechanics, Tsallis (1988) proposed independently an equivalent to Havrda-Charvat entropy function

$$S_q = \frac{C\left(1 - \sum_{i=0}^{\infty} p_i^{\,q}\right)}{q-1}$$

which was maximised subject to:

1. $\sum_{i=1}^{W} p_i = 1$
2. $\sum_{i=1}^{W} \varepsilon_i p_i = U_q$

where W is the no. of microscopic configurations and $\{\varepsilon_i, U_q\}$ are known as *generalized spectrum* and *generalized internal energy.*

- Maximisation of **S_q** gives a **Zipf-Mandelbrot** power-type distribution with non-extensive properties.

Tsallis (1988) Solution

- Introduce ***α*** and ***β*** Lagrange multipliers and define the quantity

$$\phi_q = \frac{S_q}{C} - \alpha \sum_{i=1}^{W} p_i - \alpha\beta(q-1)\sum_{i=1}^{W} \varepsilon_i p_i$$

by taking $\frac{\partial \phi_q}{\partial p_i} = 0,$ one obtains $p_i = \frac{\left[1-\beta(q-1)\varepsilon_i\right]^{\frac{1}{q-1}}}{Z_q}$

where $Z_q = \sum_{i=1}^{W}\left[1-\beta(q-1)\varepsilon_i\right]^{\frac{1}{q-1}}$

At the ***q*** → **1** limit,

$$p_i = e^{-\beta\varepsilon_i} \big/ Z \text{ , with } Z = \sum_{i=1}^{W} e^{-\beta\varepsilon_i}$$

i.e., solution of M/M/1 queue [Assi 2000, Kouvatsos and Assi 2002, Karmeshu & Sharma 2005]

G/G/1 Queue: An EME Framework

- Maximise Generalised Entropy Functional

$$\max_{P}\left\{S_q = \frac{C\left(1-\sum_{i=0}^{\infty} p(n)^q\right)}{q-1}\right\}$$

subject to

- The normalization, $\sum_{n=0}^{\infty} p(n) = 1$
- The mean queue length, $\sum_{n=0}^{\infty} np(n) = L$
- The utilisation, $\sum_{n=0}^{\infty} h(n)p(n) = 1 - p(0) = \rho,\qquad \rho = \frac{\lambda}{\mu},\ \ 0 < \rho < 1$

where *h(n) = 1*, if *n = 0* or otherwise.

Apply the Method of Lagrange's Undetermined Multipliers
[Assi 2000], [Kouvatsos & Assi 2002, 2007]

A **Generalised Zipf-Mandelbrot** EME power-type distribution

$$p(n)=\frac{\left[1+\alpha(1-q)n+\beta(1-q)h(n)\right]^{\frac{1}{q-1}}}{\sum_{n=0}^{\infty}\left[1+\alpha(1-q)n+\beta(1-q)h(n)\right]^{\frac{1}{q-1}}},\quad n=0,1,\ldots$$

- At the ***q*** $\rightarrow$ **1** limit,

$$p(n)=\frac{e^{-\lambda n-\beta h(n)}}{Z}=\frac{x^n g^{h(n)}}{Z},\quad \text{with}\quad Z=\sum_{n=0}^{\infty}x^n g^{h(n)},\ x=e^{-\lambda},\ g=e^{-\beta}$$

➔ME state probability distribution of a stable *G/G/1* queue

- *x* and *g* are the Lagrangian coefficients corresponding to mql and server utilisation constraints. Moreover, ½<q<1.

G/G/1/N Queue: An EME Framework

- Maximise Generalised Entropy Functional

$$\max_{P}\left\{S_q = \frac{C\left(1-\sum_{i=0}^{\infty} p_N(n)^q\right)}{q-1}\right\},$$

subject to:

- The normalization, $\sum_{n=0}^{N} p_N(n) = 1$
- The mean queue length, $\sum_{n=0}^{N} n p_N(n) = L_N$
- The utilisation, $\sum_{n=0}^{N} h(n)p(n) = 1 - p_N(0) = U$
- Full buffer state probability, $\sum_{n=0}^{N} s(n)p_N(n) = \varphi = p_N(n),\ 0 < \varphi < 1$

where *h(n)* = *1*, if *n* = *0* or *0* ow. and *s(n)*=*1*, if *n*=*N* or *0* ow, satisfying the flow balance condition: $\lambda\,(1-\pi) = \mu(1-P_N(0)$
c.f., *[Assi 2000, Kouvatsos & Assi 2002, 2007]*.

$G/G/1/N$ Queue : An EME Framework

- A **Truncated Generalised Zipf-Mandelbrot** EME power-type distribution

$$p_N(n) = \frac{\left[1+\alpha(1-q)n+\beta(1-q)h(n)+\gamma(1-q)s(n)\right]^{\frac{1}{q-1}}}{\sum_{n=0}^{N}\left[1+\alpha(1-q)n+\beta(1-q)h(n)+\gamma(1-q)s(n)\right]^{\frac{1}{q-1}}}$$

- At the $q \to 1$ limit,

$$p_N(n) = \frac{e^{-\alpha n-\beta h(n)-\gamma s(n)}}{\sum_{n=0}^{N} e^{-\alpha n-\beta h(n)-\gamma s(n)}} = \frac{x^n g^{h(n)} y^{s(n)}}{\sum_{n=0}^{N} x^n g^{h(n)} y^{s(n)}}, \quad n = 0,1,\ldots N$$

$$x = e^{-\alpha},\ g = e^{-\beta},\ y = e^{-\gamma}$$

This is the corresponding known solution of a GE/GE/1/K queue.
For $q<1$ and for large number of jobs $\boldsymbol{n}$, the EME solution follows the power law: $p_N(n) \sim n^{1/(q-1)},\ 1/2 < q < 1$

Boundary Conditions and a Heuristic Relationship between q and H

- A heuristic relation between the non-extensivity parameter, ***q*** and the Hurst parameter, *H* can be achieved by using the boundary conditions
- The boundary conditions of the non-extensivity parameter ***q*** of Tsallis entropy solution is $\frac{1}{2} < q < 1$;
- The boundary conditions of Hurst parameter ***H*** of the fractional Brownian Motion (fBm) is $\frac{1}{2} < H < 1$;
- It is implied that for $q \to 1$ (Shannon's Entropy) ➔ $H \to 0.5$ (exponential distribution)
- for $q \to 1/2$ (max value of non extensivity parameter)➔ $H \to 1$ (Pareto distribution with power law tails corresponding to max value of *H*)
- The following simple heuristic relationship is usually defined

$$H=1.5-q$$

[Karmeshu & Sharma 2005], [Kouvatsos & Assi 2007]

An EME Mean Queue Length

- In the context of the EME approach, a mean queue length constraint for a fBM/M/1 queue (c.f., *[Karmeshu and Sharma 2005], [Kouvatsos & Assi 2007]*) was motivated by a reinterpretation of a formula proposed in Norros [1994] in the context of ATM networks, for calculating buffer capacity of a simple storage model with self-similar input traffic process modelled by a fBm as an input process with Hurst parameter, H, $H \in [0.5,1]$ and exponential service time :

$$< n >= \frac{\rho^{1/[2(1-H)]}}{(1-\rho)^{H/(1-H)}}, \quad 0.5 < H < 1, \quad \rho = \lambda/\mu$$

where ***λ*** and ***μ*** are the mean arrival and service rates, respectively.

A Heuristic Generalisation for an EME Mean Queue Length

- A heuristic extension of Norros formula [Norros 1994] was conjectured in *[Kouvatsos & Assi 2007]* for calculating the buffer capacity of a simple storage model with generalised fractional Brownian motion (gfBm) process as an input traffic and GE-type service time distribution, namely

$$< n > \; = \; \frac{\rho^{1/[2(1-H)]}}{2^{1/[2(1-H)]}} \left(\frac{\left(1 - \rho \; + \; C_a^2 \; + \; \rho \, C_s^2\right)^{1/[2(1-H)]}}{\left(1 - \rho\right)^{H/(1-H)}} \right), \qquad 0.5 < H < 1 \, , \quad \rho \; = \; \frac{\lambda}{\mu}$$

  where *C^2a* and *C^2s* are the interarrival time and service time SCV and H is the Hurst parameter taking values in the interval [1/2, 1].

- For the computational implementation of the EME solutions, the generalised formula is adopted as the mql, *E(N),* of a stable infinite capacity gfBm/GE/1 queue.

- For *H = ½,* it yields the result for mean queue length of a stable *GE/GE/1* queue which corresponds to the case *q→1* in the proposed framework.

Overflow Probability

- The probability distribution for the queue length distribution $\{p_N(n), n=0,1,\ldots\}$ can be rewritten in term of Hurwitz-Zeta function as,

$$p_K(n) = \frac{\left[\dfrac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}+n\right]^{\frac{1}{q-1}}}{\varsigma\left[\dfrac{1}{1-q},\dfrac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right]},\ \{q>0,\ n=0,1,\ldots.K\}$$

- the overflow probability,

$$P(n>x) = \left(1\Big/\varsigma\left[\frac{1}{1-q},\frac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right]\right)\left(\frac{1-q}{q}\right)\left[x+\frac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right]^{\frac{-q}{1-q}}$$

$$-\left(1\Big/\varsigma\left[\frac{1}{1-q},\frac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right]\right)\left(\frac{1-q}{q}\right)\sum_{n=x}^{N}\int_0^1 u.\left(u+n+\frac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right)^{-(2-q)/(1-q)}du$$

Overflow Probability

- For asymptotically large ***x*** a power law is determined by,

$$P(n>x)\sim Wx^{-q/(1-q)},\quad W=\left(1\Big/\varsigma\left[\frac{1}{1-q},\frac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right]\right)\left(\frac{1-q}{q}\right)$$

- at the ***q* → 1** limit**,**

$$P(n>x)\ \sim e^{-\alpha x-\beta h(n)-\gamma s(n)}$$

Server Utilisation and Blocking Probability

- The probability that server is busy (i.e., the server utilization, *U*)

$$U = 1 - p_K(0)$$

$$= 1 - \frac{\left[\alpha(1-q) + \beta(1-q)h(n) + \gamma(1-q)s(n)\right]^{\frac{1}{q-1}}}{\varsigma\left[\frac{1}{1-q}, \frac{1+\beta(1-q)h(n)+\gamma(1-q)s(n)}{\alpha(1-q)}\right]}$$

- Using the flow balance condition, the blocking probability can be obtained by

$$\pi = 1 - \mathrm{U}/\rho$$

Note: All these formulae together with the associated algorithms below can be found in *[Kouvatsos & Assi 2007]* & are generalisations to those reported in *[Karmeshu & Sharma 2005*].

EME Analytic Algorithms

EME ALGORITHM 1: The gfBm /GE/1 Queue

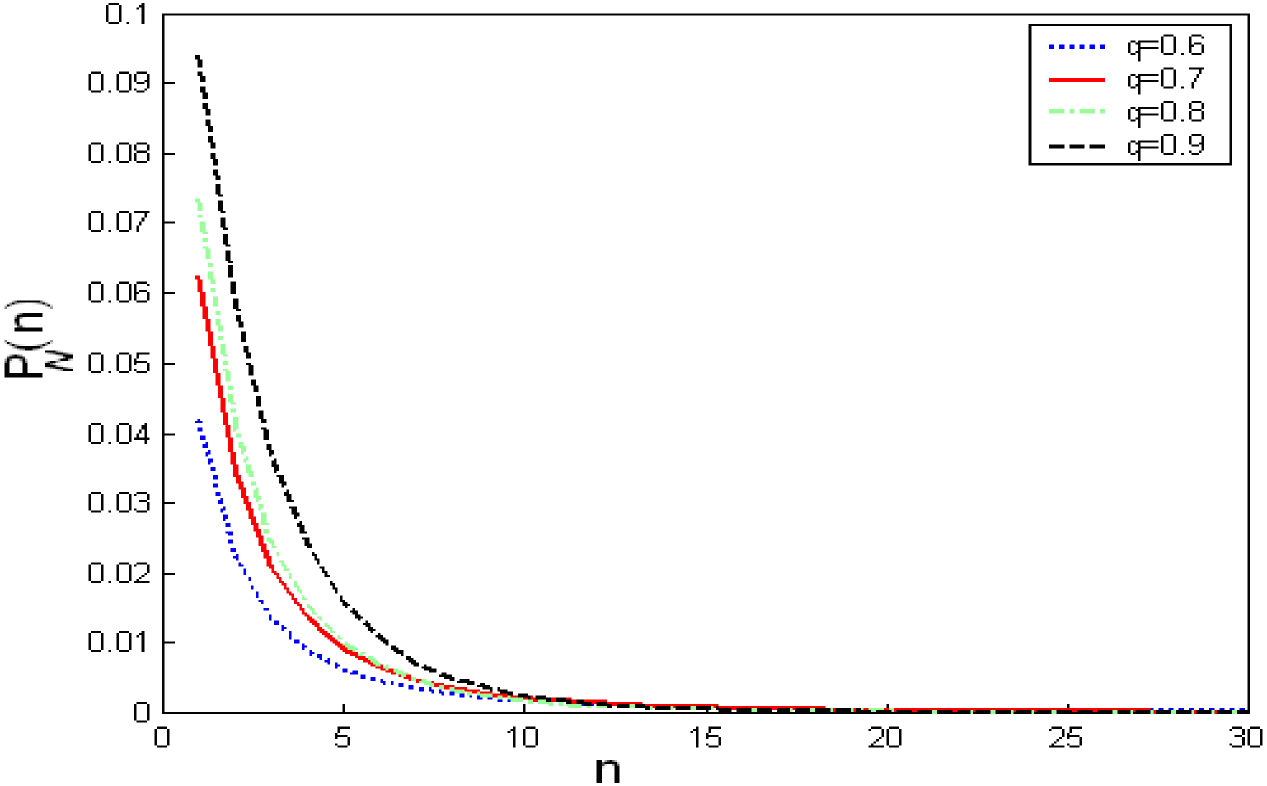
- *Input Data* $\{q, \lambda, C_a^2, \mu, C_s^2\}$
- *Begin*
- *Step 1* Calculate *H=1.5-q* and mean queue length, $<n>$
- *Step 2* Set initial approximations of Lagrangian multipliers $\{\alpha, \beta\}$;
- *Step 3* Solve constraints (2) and (3) via Newton-Raphson method wrt $\{\alpha, \beta\}$;
- *Step 4* Obtain new values for $\{\alpha, \beta\}$;
- *Step 5* Return to Step 3 until convergence of $\{\alpha, \beta\}$;
- *End*
- *Output Statistics:* The Lagrange's multipliers $\{\alpha, \beta\}$ and state probabilities, {p(n)}.

EME Analytic Algorithms (cont.)

EME ALGORITHM 2: The Censored gfBm */GE/1/N Queue*

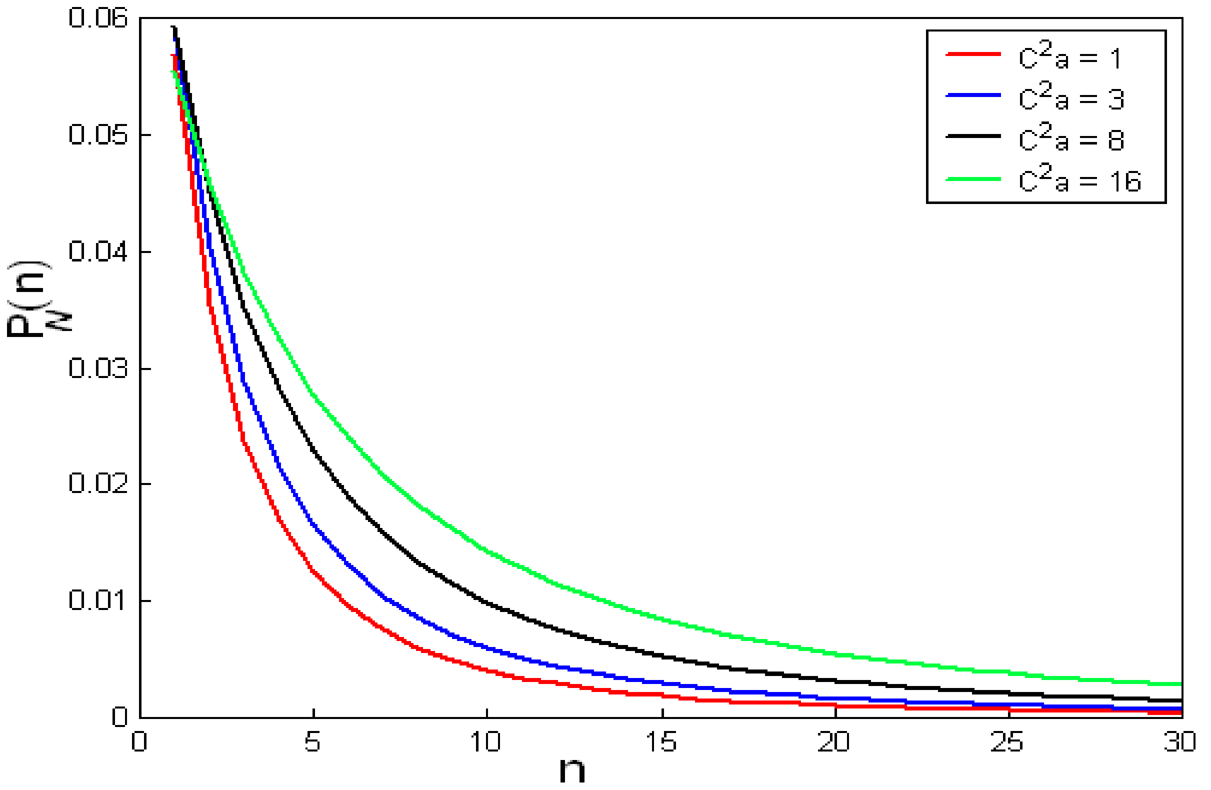
- *Input Data* { $N, \alpha, \beta, q, \lambda, C_a^2, \mu, C_s^2$ }
- *Begin*
- *Step 1* Initial approximation of Lagrangian multiplier γ ;
- *Step 2* Solve constraints (1) and (4) using the Newton-Raphson method wrt γ ;
- *Step 3* Obtain new values for γ;
- *Step 4* Return to Step 2 until convergence of γ ;
- *Step 5* Using flow balance condition to compute blocking probability.
- *End*
- *Output Statistics:* The Lagrangian multipliers, γ , state probability (P_N(n)} and the blocking probability, π

Numerical Results



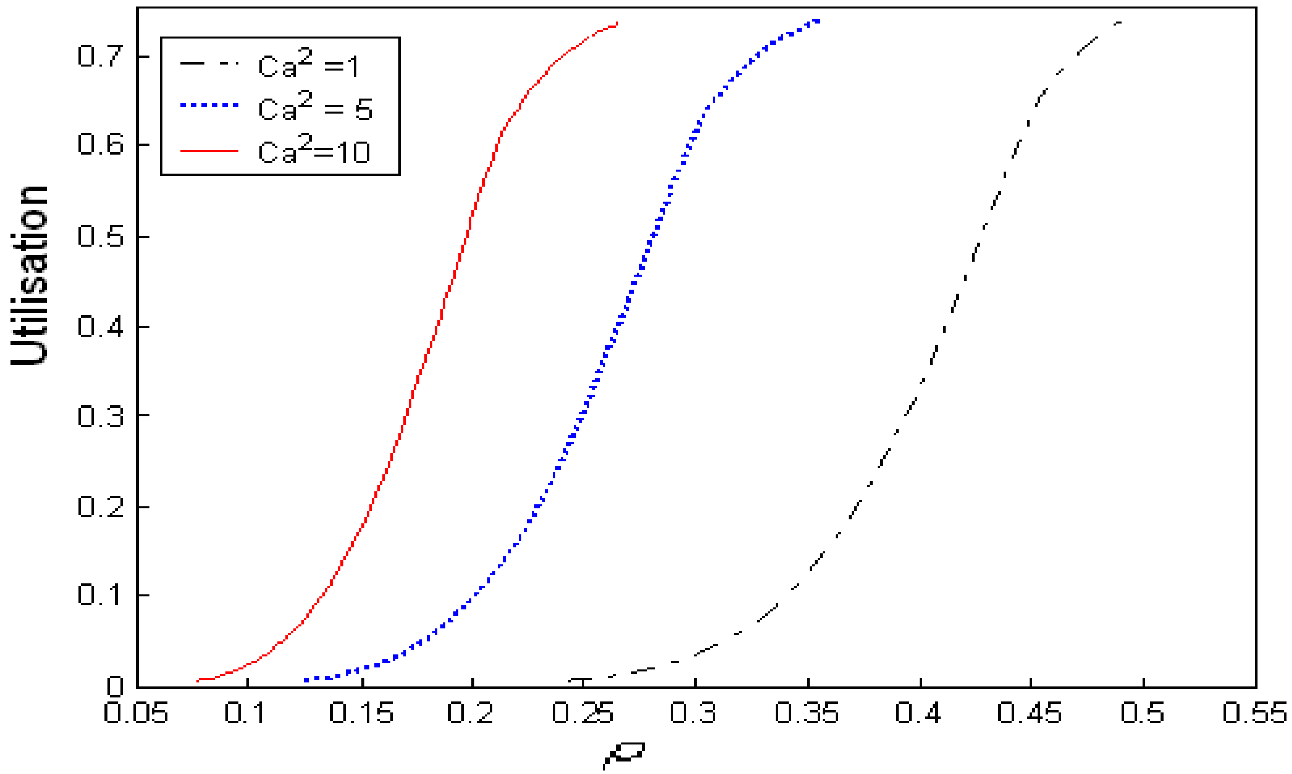
The relation between $p_N(n)$ and n for a finite capacity queue with (q = 0.5, 0.6, 0.7, 0.9) , $C^2a = 8$, $C^2s = 4$, λ=0.03, μ=0.2 and N =30.

Numerical Results



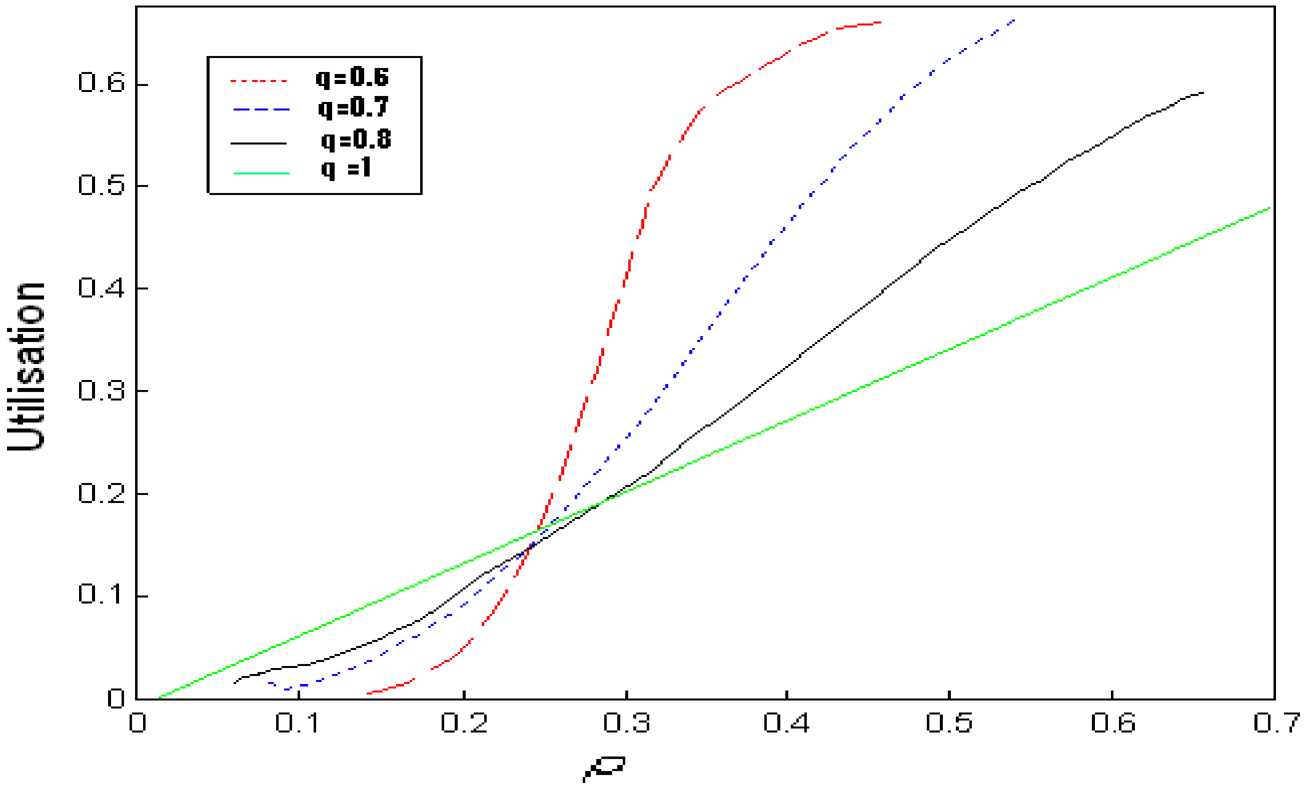
The relation between the queue length distribution and *n* for a finite capacity queue with λ=0.02, μ=0.4, q = 0.6, C^2s = 4 and N =30

Numerical Results



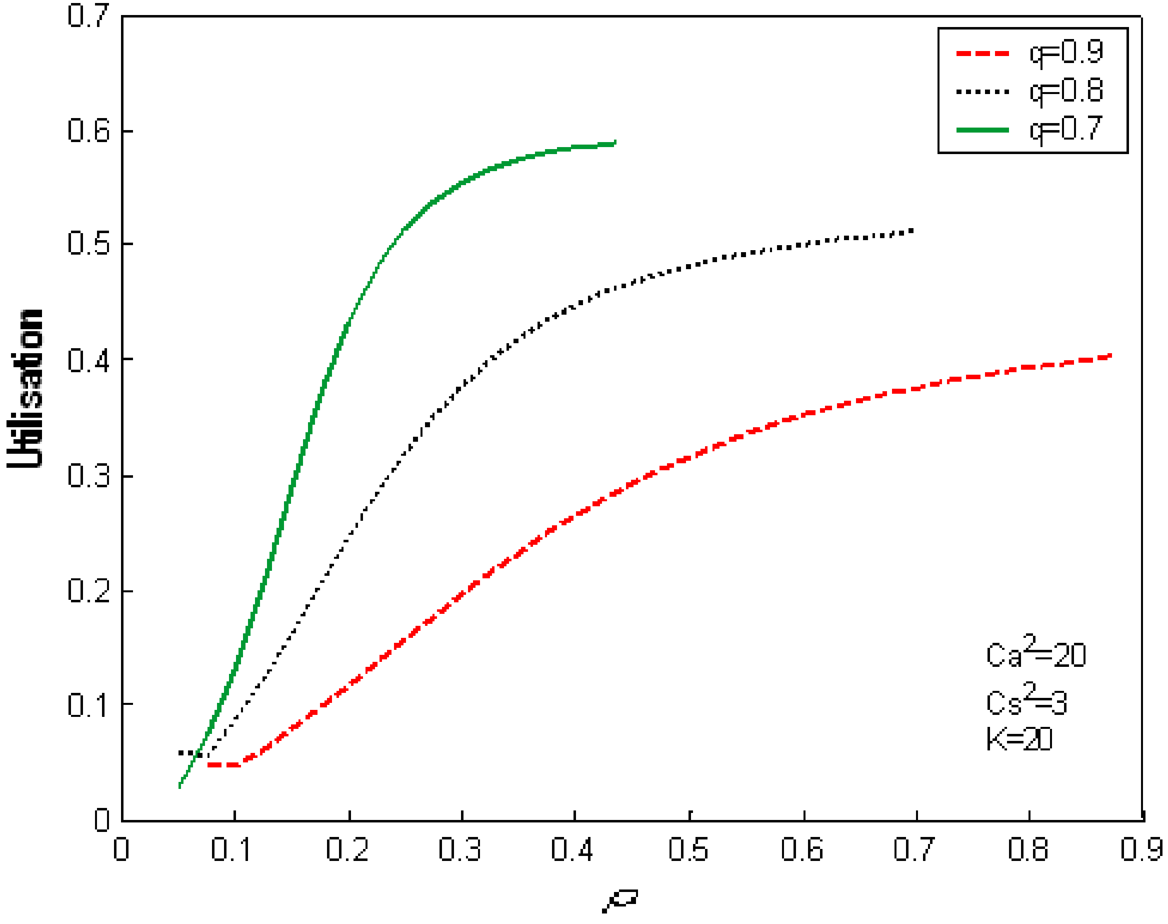
The relation between ρ and U (Utilisation) with {C^2a = 1, 5 & 10}, C^2s = 3, q =0.6 and N =20.

Numerical Results



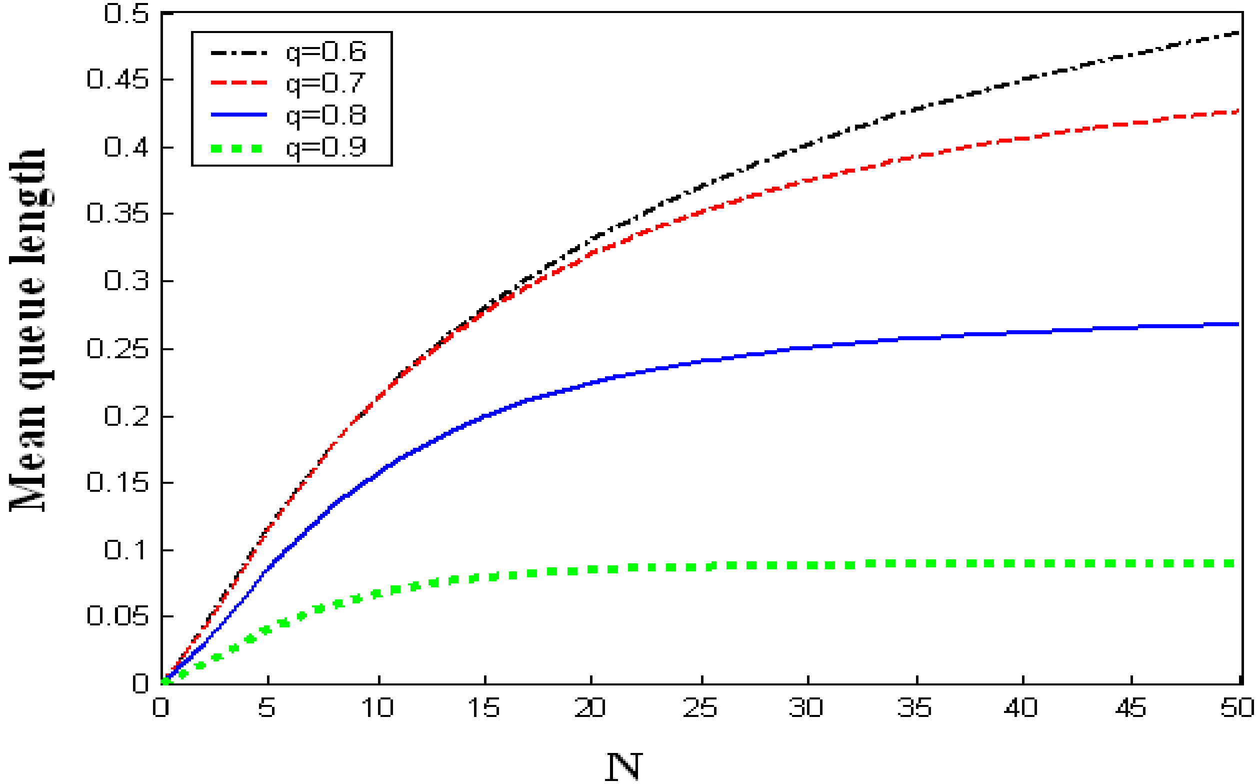
The relation between ρ and U (Utilisation) with $C^2a = 3$, $C^2s = 4$, q ={0.6,0.7,0.8,1}

Numerical Results



The relation between traffic intensity, ρ and server utilisation, U = 1- P(0) for a finite capacity GfBm/GE/1/K queue with Ca2 = 20, Cs2 = 3 and q= 0.7, 0.8, 0.9.

Numerical Results



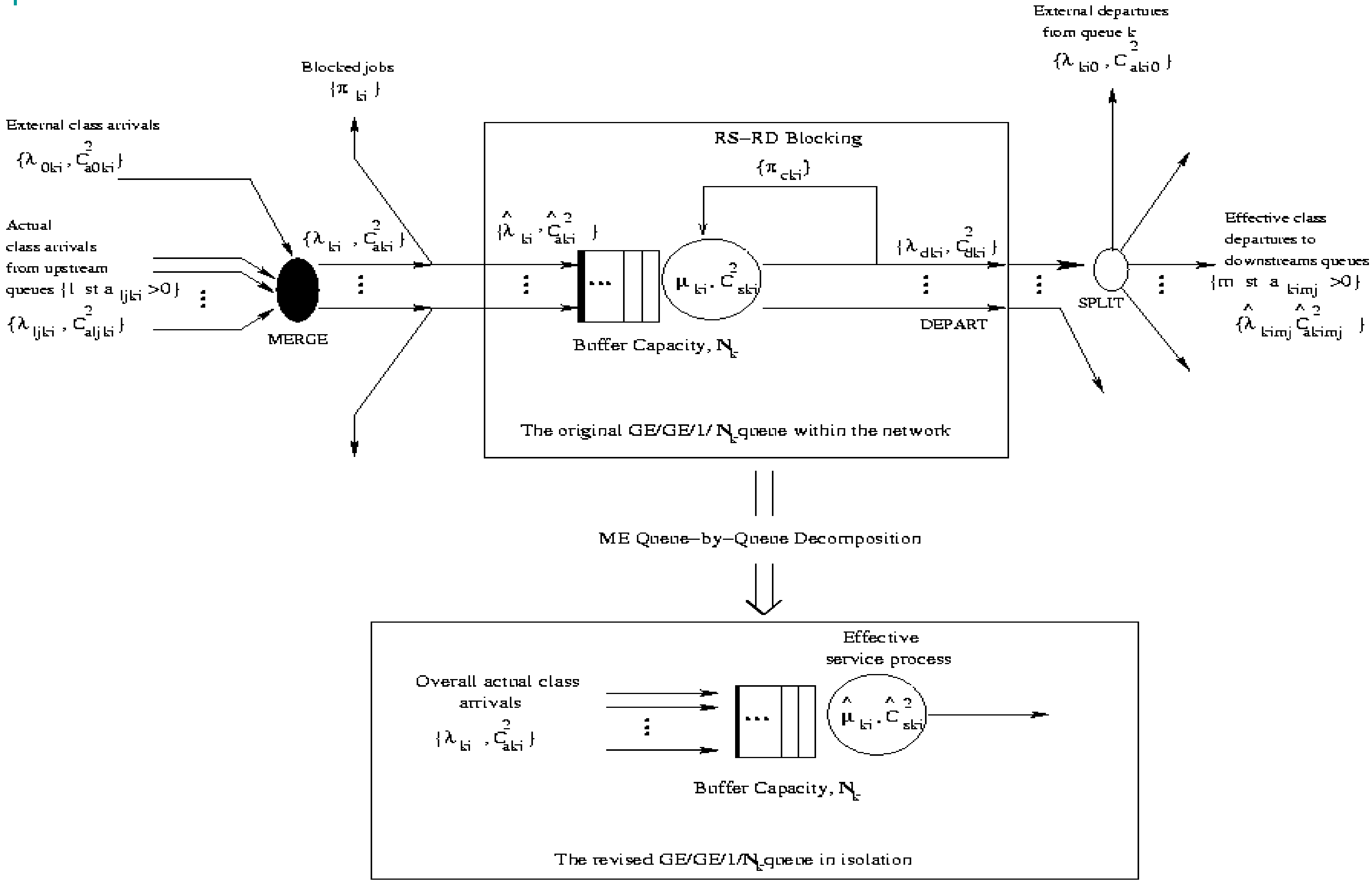
The relation between the mean queue length and N (queue capacity) with C^2a =4, C^2s = 9, λ=0.1, μ=0.5 and (q =0.6, 0.7, 0.8, 0.9)

Conclusions & Extensions to Arbitrary Open Queueing Network Models (QNMs)

- Product-Form Approximations and Queue-by-Queue Decomposition of Arbitrary Open Queueing Network Models (QNMs) with Blocking

[Kouvatsos & Awan 2003]

Queue-by-Queue Decomposition of Open QNMs



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END